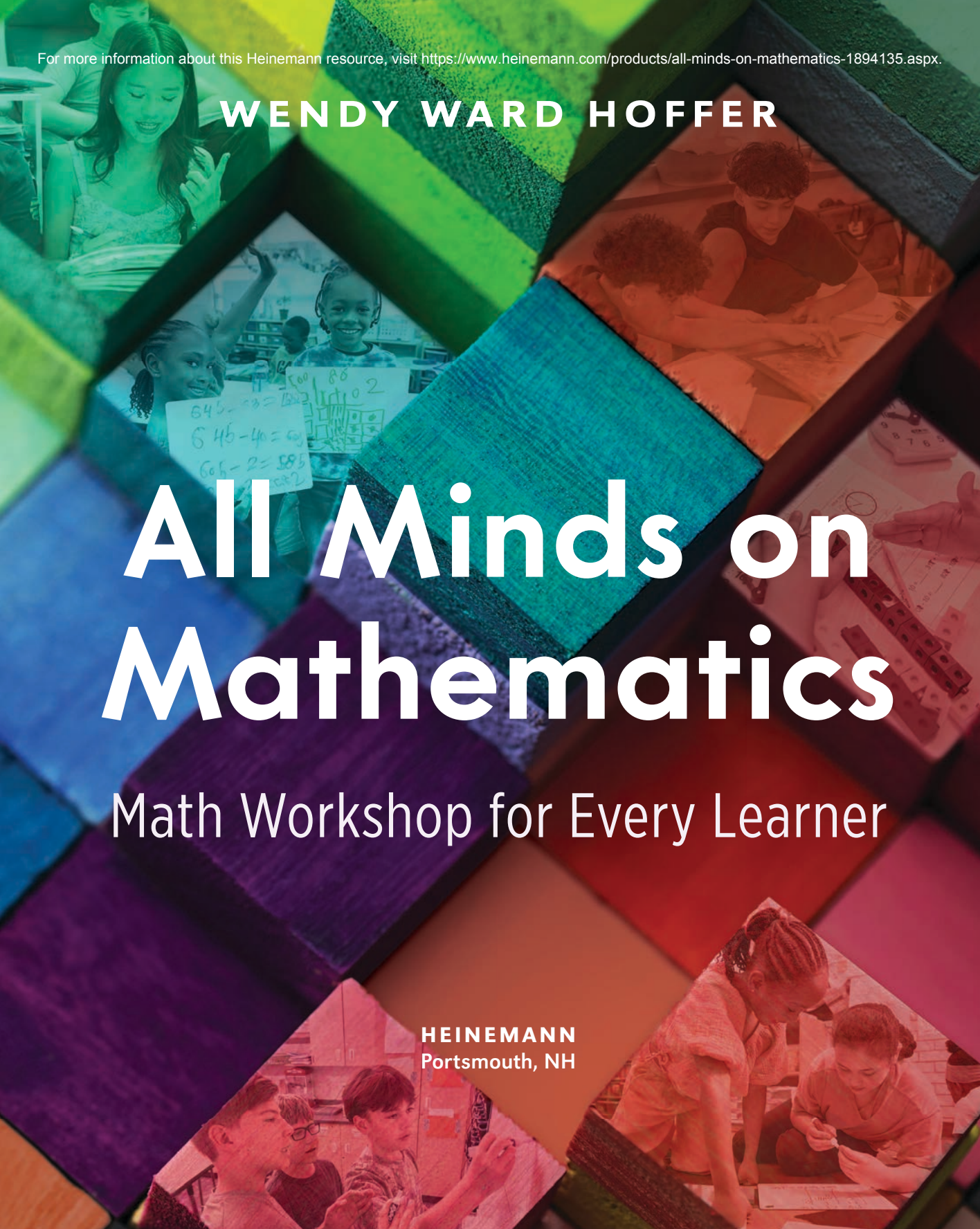
WENDY WARD HOFFER



All Minds on Mathematics

Math Workshop for Every Learner

HEINEMANN
Portsmouth, NH

Acknowledgments:
Council of Chief State School Officers. (2013, April). *InTASC Model Core Teaching Standards and Learning Progressions for Teachers 1.0*. Washington, DC.

From *Adding It Up: Helping Children Learn Mathematics*, National Research Council. National Academies Press.

"Wheel of Power/Privilege." © Sylvia Duckworth, sylviaduckworth.com. Used with permission.

Acknowledgments continue on page 189.

Library of Congress Control Number: 2025950562

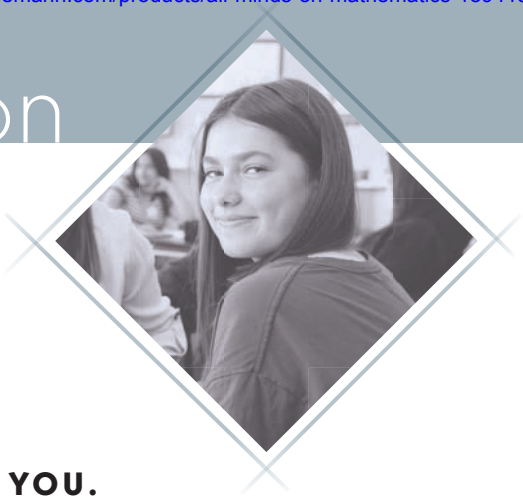
Printed in the United States of America on acid-free paper

ISBN-13: 979-8-202-06747-1

4500926795 r03.26

Contents

Introduction



"I SEE YOU, I BELIEVE IN YOU. YOU ARE SAFE TO GROW AND THRIVE HERE. I WANT TO HEAR YOUR VOICE."

—Jamila Dugan and Shane Safir

A couple of years ago, I entered an elevator in a downtown office building and reached over to press the button for the fourth floor, where I was headed. (I know, I could have taken the stairs.) A young woman dove in behind me as the door was closing and extended her hand up the tall column of buttons to the nine. Feeling chatty, as I usually do, I asked her, "What's on the ninth floor?"

"A small engineering firm," she shared shyly.

"Oh. Are you an engineer?" I asked.

"Yes," she affirmed, looking me square in the face now. "And you were my middle school teacher."

I asked for her name, as I did not recognize her all these years later, and came to realize that a once-meek seventh grader now occupied a wide glass-

walled office with a beautiful view of the city where she worked to design more of it. When we got to my floor, I stepped out thinking how this young woman had, quite literally, risen above me, her once-teacher. This is, in fact, my hope for all learners: that they each, in their own ways, rise above the generations before and reach beyond us, the families and educators who raise them.

And yet, for many of our students, even today, access to mathematical heights is limited.

Why Math Matters Most

While every content area deserves attention, mathematics is unique in its liberatory possibilities. Students with strong math skills

- demonstrate higher levels of self-confidence and agency
- identify as problem solvers with the tools and stamina to persevere through life's challenges
- gain access to advanced coursework in high school and beyond
- succeed on standardized tests that provide access to higher education and scholarship support
- earn the credentials to enter STEM jobs and other lucrative positions
- ensure international competitiveness in technology and other generative fields
- pass on math confidence to their families, advancing generations.

For these reasons, math is often identified as a "gatekeeper" subject, one with the power to open potential future pathways. Conversely, students who do not pass high school algebra by the end of ninth grade have only a one in five chance of graduating high school (Gates 2021).

Those non-graduates are not distributed equally across racial lines: the average four-year high school graduation rate nationwide for students

receiving regular diplomas in 2019 was 86 percent; yet, of that cohort, who were our graduates? Ninety-three percent of all Asian/Pacific Islander students, 89 percent of white kids, 82 percent of Hispanic learners, 80 percent of Black scholars, and 74 percent of Native and Indigenous twelfth graders graduated nationwide. By state, the Black versus white graduation gap varies from 1 percent (Hawaii) to 24 percent (District of Columbia), with a national average of 10 percent (NCES 2024).

With algebra prowess a strong predictor of overall success and opportunity, there are few efforts more economical than putting our shoulder to the bumper of the math bus.

Why *All Minds on Math*?

I wrote *Minds on Mathematics* in 2012. A middle school math and science teacher turned teacher educator, this was my second publication synthesizing research and practice in ways that were accessible for teacher audiences of all experience levels. The book's title was derived from the idea that learning needs not only to be hands-on, with math manipulatives and other experiential invitations to develop understanding, but also *minds* on, with ample opportunities for thinking and reflection amid all the action. That book presented math workshop as a forum for engaging students in rigorous math tasks aligned to the Common Core State Standards for Mathematical Practice supported by student discourse and the resource of thinking strategies. I stand by it.

Yet, the world has changed dramatically since that book's publication: We are now digitizing everything from curricula to state tests. We engaged in a reckoning about systemic inequities and are now witnessing anti-DEI policies. We navigated a global pandemic that accentuated educational inequality. We face a youth mental health crisis. Teacher turnover is at an all-time high. The administration, at the time of writing this book, is striving to undo the federal Department of Education. And, amid all this, our math scores founder.

Meanwhile, I have continued to serve as a staff developer with the Denver-based, nonprofit Public Education and Business Coalition (PEBC), steadfast in our mission to support and sustain schools and systems worthy of our children. In this role, I observe and coach teachers, read and research,

design and facilitate professional learning for educators, and write. My work as a math educator is centered on three convictions:

- Numeracy is a civil right.
- We are all capable mathematicians.
- Math is about making sense.

Across time, I have learned about accessibility, creating math workshops that serve each and every learner, regardless of ethnicity, economics, or historical achievement. This book endeavors to build on the now-familiar frame of math workshop to add nuance and insight to solve today's critical problems facing math students: their needs for safety, inspiration, challenge, support, growth, and, above all, joy, in their learning experiences.

Minds on means that learners are actively engaged as thinkers throughout the lesson, not passively watching or mindlessly poking at screens. This book is titled *All Minds on Mathematics* because it is at once an extension of the ideas presented in the earlier text and an invitation to be more inclusive by involving learners of all backgrounds in the important work of thinking as mathematicians, and it contains recommendations to ignite more aspects of each learner's mind—their memories, feelings, and identities, along with their logical problem-solving prowess. The purpose of this book is to create a vision for effective mathematics instruction, as well as to provide teachers with the stances, strategies, tools, and skills needed to actualize high-quality learning experiences for all students.

My Beliefs

When I summarized my beliefs about the importance of math instruction recently to my state's House Education Committee in support of new legislation to fund math initiatives, Colorado State Representative Jennifer Bacon asked what it would look like to provide better math instruction. I bumbled to put a career of research and practice into the brief three minutes allotted for my reply, noting my own clumsiness in the rush. In this book, I seek to take the time to respond to her very important question in greater detail, for the benefit of a broader audience.

As educators, we have the transformational power to

- believe in every student's right to an excellent math education
- cultivate learners' perseverance, agency, effort, and joy in math and beyond
- provide rich, affirming opportunities for mathematical thinking
- challenge learners in a manner that honors and extends their understanding
- celebrate diverse ways of knowing
- pursue excellence on all measures of achievement.

These efforts are the essence of *All Minds on Mathematics*.

Teaching and learning math are acts of hope. When we choose to teach math and to teach it well, we say to our students, "I believe in you and your great potential." Isn't that what our children need to hear right now? Ultimately, this work goes far beyond our classrooms and extends to our duty as adults to raise thoughtful, reflective, collaborative young people able to see beyond their individual points of view and combine efforts to solve the many problems their lives—and our world—will present.

About This Book

This text pulls together three important fields of study: effective math pedagogy, culturally responsive teaching, and social-emotional, trauma-informed practice. It also draws on PEBC's four decades of work supporting schools and systems in honoring and uplifting children's innate capacity as thinkers. The first chapters lay out a vision for our students as capable mathematicians (Chapter 1) and then a vision for ourselves as math educators focused on our own and our students' assets (Chapter 2). The balance of the text is devoted to articulating the design and facilitation of effective math workshops to those ends, chunk by chunk, named by purpose: Safety, Inspiration, Challenge, Support, and Growth. Each chapter includes some research, some implementation ideas, and true classroom vignettes culled from my work in schools (with the students' names changed to protect their privacy).

Reflection and synthesis questions are offered throughout as a means to spark your consideration or discussion; please don't skip them; their purpose is to engage you more deeply in integrating the ideas presented.

All the vignettes in this book are true, and each line of classroom script is authentic. Anyone who has invited me to observe a lesson knows that I always bring my laptop and sit typing as quickly as I can while teachers and learners are at work, capturing actual conversations word by word. If at some point in your reading you find yourself asking whether these classroom stories can be real, I assure you that they are—students are indeed capable of such thoughtful work and clear insight—and I am ever grateful to my esteemed colleagues who welcome me in to witness their and their students' brilliance year after year. My goal in recording these stories is to inspire you to expect even more from learners.

While the majority of the examples provided here are from upper elementary through first year high school classrooms, my experience and collaboration with K–12 colleagues has taught me that these methods, adjusted, can be used with learners of all ages, including adults. I invite you, as you read vignettes, ideas, and examples from grade levels and topics other than those you teach, to pause and sit back, and ask yourself at each turn, "What might be the implications for my own work?" In this way, you will reap the greatest benefit from your reading.

While this book can certainly be studied and implemented by individual educators, my hope is that you will find a colleague—or, better yet, a team, department, or entire faculty—ready to engage with you in reading and discussing the questions offered at the end of each chapter, and, most importantly, in experimenting with the ideas proposed here. In these ways, you can inspire and support one another through the challenges of engaging more and more and more *and more* minds on math.

As you dive into this journey, I offer you one last recommendation: you do you. This book is full of recommendations, research, examples, resources, all of which I invite you to pick up, turn over in your mind, try on, adjust, test drive, and discern, "Does this work for me and my students, given our context?" Mess about with these ideas. Give yourself permission to experiment, grapple, problem solve, and try again. Your math workshops will evolve to be a unique and beautiful reflection of your own strengths and vision. *There is no wrong way to do math workshop*, I like to say; if students are thinking as mathematicians, you are doing it right!

Synthesis Questions

- Why do you value math teaching and learning?
- Which students do you feel are being well served by your current math instruction? What are your beliefs about why this is the case?
- What are you already doing to get more learners' minds on math?
- Of the bulleted invitations to educators listed on page xiv, which speak to you and how?

1

PURPOSE

Problem: Americans are frenemies with math.

Solution: As math educators, we can inspire learners as capable mathematicians.

When I mention my work alongside math teachers and math learners to people unfamiliar with my job, they often have a visceral reaction—everything from, "Oh, I hate math!" to "I wish I'd had a math teacher like you." Some share their memories of being math students, many unfortunate. Math-weary adults recall being ignored or lost in math class, their sense of not belonging. And parents bemoan their own ineptitude at helping their children with math. After more than twenty-five years dropping the topic of math teaching into friendly conversation, my qualitative research suggests that Americans have a problem: We are simply not a math country—yet. One look at international comparative math achievement data reinforces

that assessment. Although there are some strivers who breeze through the sequence of math learning to calculus and beyond, too many people disdain math and perceive it as an annoying impediment. As a nation, and as educators, math is our growing edge.

Meanwhile, we in the U.S. are nevertheless hell-bent on using math as a measuring stick, a sorting tool, evidence of a student's potential, and for decisions about their opportunities. Math is an on-ramp, a skill that can open doors to advanced courses, university acceptance, scholarships, internships, and jobs. And yet, opportunities to thrive in math are not equally available to all students. In the book *Radical Equations*, civil rights activist and math educator Robert Moses wrote,

> *The most urgent social issue affecting poor people and people of color is economic access. In today's world, economic access and full citizenship depend crucially on math and science literacy.* (Moses and Cobb 2002, 5)

Mathematical prowess is a power tool. It can bore through historical inequities and lay the foundation for students' success in their careers and intellectual endeavors. Each and every student needs confidence, opportunity, and support to learn this superpower. As described by mathematics professor and author Francis Su, "Not tapping everyone's potential is a loss for all of us and will limit the ability of future generations to solve the problems they will face" (2020, 7).

Reflection
What are your greatest hopes for your math learners?

To unleash the mathematical potential of all learners, we need to begin with a clear vision: What exactly are we trying to accomplish?

Your Math Story

What is your math story? Who inspired you, or shook your confidence? When have you felt like a mathematical victor? An algebraic imposter?

In the space below, take some time to reflect: Create a line graph of your confidence as a mathematician, starting in childhood. Note inflection points across your learning experiences. What were moments that sent your confidence soaring? Which experiences caused you to falter or doubt yourself?

And for how long? Where do you stand now? Our own experiences as math learners, as well as our conscious reflection on the impacts of those experiences, inform our perspectives about effective math teaching.

Reflection
How does your personal history with math inform your instruction?

Capable Mathematicians

All our students can be math people; agentic lifelong learners who relish the grapple of mathematical meaning making. We have the opportunity to guide each student in their efforts toward becoming capable mathematicians who possess the following:

- productive dispositions
- number sense
- conceptual understanding
- problem-solving capabilities.

These are not distinct abilities but instead they flow across one another and can each be honed with conscientious effort. Rather than teaching them in isolation as distinct parts of a lesson, we can integrate them throughout an effectively designed and facilitated math workshop, as will be described throughout this book.

With these skills, students will be prepared to handle the challenges—mathematical and otherwise—that their lives may throw at them. And

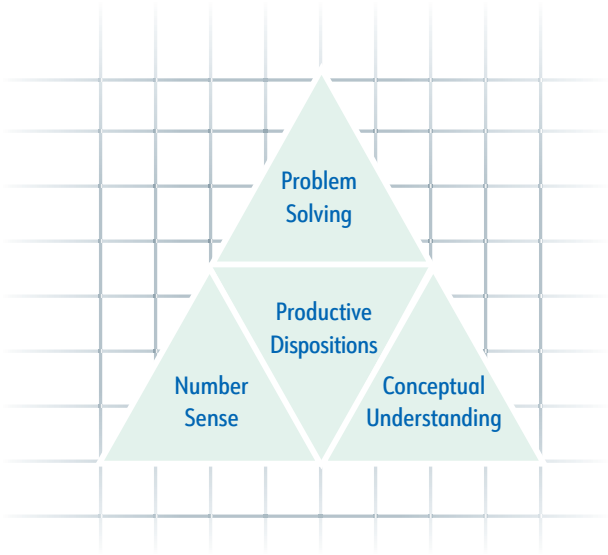


Figure 1.1 Features of Capable Mathematicians

(Adapted from Kilpatrick, Swafford, and Findell 2001)

decades from now, looking back on their own math learning experiences, your students will smile to themselves remembering you as that teacher who believed in their potential to do hard things, and how you inspired, challenged, supported, and then gave them room to solve complex problems for themselves.

Let's explore how to instill these abilities in our students.

Productive Dispositions

A productive disposition is a no-holds-barred stance of possibility. Think about six-year-old Ruby Bridges marching up the front stairs of William Frantz Elementary School despite its history of segregation, or firefighters charging into the burning World Trade Center towers on 9/11, each with determination and confidence. A student with a productive disposition understands that math is relevant, meaningful, and accessible, and perseveres in their efforts to "Make sense of problems and persevere in solving them" (Common Core State Standards Initiative n.d., 6).

By the time they reach us, too many students have decided they're not strong in math. Maybe their parents or siblings threw their game, or a previous teacher passed along a fear or bias. We may never know what rattled their confidence, but it is never too late to build their identities as problem solvers who believe in their own competence. As Stanford math education expert Jo Boaler describes in *Mathematical Mindsets*, "No one is born knowing math, and no one is born lacking the ability to learn math" (2022, 6).

Cultivate Productive Beliefs

We can each model a math-positive stance, frequently sharing our own high regard for math and reporting on all the ways math helps us navigate our lives. I often invite teachers to notice and tell students stories of our own mathematical exploits: how we calculate a tip, measure mileage on a trip, act on an investment tip, decide how much to chip in for a meal—and more—are all matters resolved with math. Further, we can also each report on times when we grappled and persevered as mathematicians and otherwise, illuminating this virtuous quality of character. Catch yourself as a persevering mathematician, and then tell the kids!

In addition to sharing stories, we can facilitate discussion inviting students to let go of unhelpful ideas and cultivate productive beliefs about math and themselves. The following core ideas are essential to their mathematical success:

- Math is valuable.
- I am capable.
- Perseverance prevails.

We can impart these beliefs through narration and discussion all year long within the context of our math workshops. Here are some conversations to host:

WHY IS MATH VALUABLE?

Invite your students to think of 100 ways math skills can improve their lives. Maybe start with practical applications such as ways of saving money to take a vacation. And maybe discuss concepts like compound interest so they can avoid getting trapped in credit card debt. Also explore the academic and career benefits of developing math skills, including access to challenging courses, scholarship opportunities, and job offers. Remember to discuss math as a logical beauty, a divine design that can provide meaning and order in the context of an otherwise chaotic world. The infinite patterns of snowflakes, for example, or the ways M.C. Escher used art as visual metaphors for mathematical concepts can inspire these conversations.

IN WHAT WAYS ARE WE CAPABLE?

Some learners may doubt their own prospects as mathematicians. Here you might model your own humility by telling your "math story," and leaning in

with them, noticing and naming their acumen as problem solvers: "Look at you track down your missing coat!" or "I see you made space on the rug for your friend!" These problem-solving skills are innate to each of us. In math, we apply them to look at a challenge, use our resources, conjure a possible approach, and persevere until we prevail. It is not always easy or quick or even pleasant, but this is the work—to dig deep and make effort—and we are each capable of doing it! Whenever possible, catch students in acts of problem solving; by noticing and celebrating their skills, you help them embrace their identities as mathematicians.

HOW DOES PERSEVERANCE SERVE US?

The human race has a history of perseverance. We are each here because our ancestors survived through challenging times—perhaps drought or famine or oppression—to reproduce and raise their children. We have in us the genetic blueprint for determination and can apply that attitude to any goal we set. Psychologist Carol Dweck's (2006) research on growth mindset demonstrates that effort, more than innate ability, makes the most difference in any field of endeavor.

Honor Growth

Our feedback impacts learners' ideas about themselves as mathematicians. If we focus on celebrating right answers and quickness, we promote the notion that math talent is inherited and that fast, correct answers are the only manifestations of success. That approach leaves little room for the strivers. The quick, right kids already, for the most part, know they are quick and right and tend to believe that their smarts are innate, without needing to be challenged or stretched. Instead, let us cultivate in our students an appreciation that we can all improve through determination. To this end, our growth-focused praise can highlight learners' diligence. Look for chances to celebrate your students' growth mindsets and effort with phrases such as:

- "You stuck with it!"
- "You changed your thinking!"
- "You found another way!"
- "Wow! Your effort really made a difference."

Catching students in the act of persevering reminds other learners how persistence looks: Congratulate a kid for leveraging an apt tool. Laud the furrowed brow of another intent on working through a problem. Toot the horn of a student who had a hard-earned breakthrough in understanding a difficult concept. Delight in learners embodying productive beliefs. When we praise and honor growth, we imbue in students an appreciation that their determination is of value and inspire more of it. Learners are listening.

Celebrate Diverse Ways of Knowing

Some problem solvers like to write out their thinking as they puzzle through math problems. Some draw models or write their steps on paper, while others work in their heads. Some arrive quickly at answers, while their classmates may take more time. Regardless of how we're disposed to make meaning, we are all indeed mathematicians.

Reflection

In what ways are you deliberately cultivating productive dispositions in math learners?

Everyone's mind is different. While for many math problems there is one and only one correct answer, the possibilities for solution pathways are often endless. When we offer space and time for learners to think creatively and represent their ideas in their own ways, we offer opportunities for each to deepen their understanding by synthesizing a range of possible approaches. Whether learners are neurodivergent, multilingual, physically challenged, or considered "mainstream," we have the chance to uplift their diverse thinking and support their productive beliefs about themselves and their own capacities.

How we invite and facilitate exploration of mathematical ideas communicates what we value. Let's listen in as elementary teacher Holcomb Mosley welcomes learners to consider what may appear to be a very straightforward problem: One basket of strawberries costs $3, so how much would two baskets cost? She invites one student to share his approach, not just an answer.

"Two times three is six."

"Why two times three instead of three times two?" Ms. Mosley asks.

"There's not three baskets, there's two, and it's not two dollars, it's two baskets."
"Do you have two equal groups of three?"
"Yes."
"How much would four baskets cost?" she asks another student.
"Twelve."
"How do you know?"
"Four times three is twelve. I figured it out by knowing four plus four is eight, and then I added another four to get twelve."
Another student shares: "I knew that two times three is six, so I just did four times three equals twelve."
"Does it work every time—that if you double one factor, the product also doubles?"
"No," some students chime in.
"Let's try it out."
A boy sitting up front tries, "Two times five equals ten, so four times five equals twenty."
Ms. Mosley narrates, "You doubled the factor, which doubled the product. Do you think it works every time?"
Students are unsure. "No," some murmur.
"Turn and talk to your partner," the teacher instructs. "Can you think of a time when it doesn't work?"

The discussion continues between partners, as they lean over small whiteboards and try out various multiplications. In this way, Ms. Mosley welcomes learners to conjure and solve their own problems to determine whether the conjecture holds. As capable mathematicians, they get to figure it out for themselves and hear various ways of knowing in the process.

Because there is no "wrong" way to "see" a quantity, questions of this nature offer open-ended invitations, welcome creativity, prompt productive discussions, and inspire insight. Our curricular materials often offer set models of concepts, which can be helpful, but should be used only to facilitate understanding, not to restrict the unlimited variety of ways that understanding can be expressed. It is often those who can "see" in a unique manner that open the doors to unforeseen solutions. Let us invite this input and celebrate these abilities!

Number Sense

Another aspect of becoming a capable mathematician is the development of number sense: the ability to understand and work with numerals, grasp them and number systems as abstract representations of concrete quantities, and know how to compare and combine numbers in various ways. This is but one aspect of math, not its entirety. When we have number sense, we are able to think flexibly about numbers and how to compose and decompose them with a variety of operations.

Teaching number sense involves exposing students to many opportunities to be creative and versatile. Number talks in which learners explore various ways to represent a quantity, number sense routines that invite students to think and talk about math, whether estimating or inferring, and other quick, daily activities can be useful tools for students to rehearse their acuity with numbers. These invitations can work well at the opening or closing of a math workshop, or in those few minutes we find at school between one scheduled event and another.

Concrete—Representational—Abstract

For many of us versed in the language of math, the number four, for example, possesses a world of meaning: the number of corners of a square, bases on a baseball diamond, or wheels on a typical car. Our ability to see the symbol "4" and pull up these mental models is evidence that we are able to move fluidly from the abstraction of that symbol to envision many concrete examples of that quantity. Likely, we can each also create a variety of representations of four—four tally marks, dots, or stars, for example. In this way, we demonstrate our number sense about this value. As noted in the National Academy of Sciences publication, *Adding It Up*:

> *All mathematical ideas require representations, and their usefulness is enhanced through multiple representations. Because each representation has its advantages and disadvantages, one must be able to choose and translate among representations.* (Kilpatrick et al. 2001, 2)

While some of our students walk in with the dexterity to translate from the concrete to the representational to the abstract and back again, we can intentionally foster this skill among all learners with regular practice.

Moving between these models is also a go-to strategy for addressing confusion when it arises. "How do you see that?" I might ask a learner about a solution, or "How might this be represented?" With questions of this nature, we remind learners to conjure the actual value signified by symbols rather than the signifiers themselves. In this way, we continuously pull students back to the meaning of a problem and the work it calls us to do.

Manipulatives are fantastically useful at every level for developing number sense. They offer students a tactile representation of quantities and operations, more tangible than images on a screen or sketches on paper. If you or your school have shied away from concrete manipulatives, please dig them out, dust them off, organize them for access, and teach students how to use and manage these tools. They are invaluable resources for building numeracy.

Practice

Developing number sense takes time. Number talks and number sense routines are common rituals for engaging learners in regular numeracy practice. They may involve interpreting a visual model, representing a quantity in a variety of ways, or narrating various approaches to an equation. The goal is

to invite students to understand the concrete value of numbers and to think flexibly about how numbers might be expressed.

Some curricula provide a variety of math games tagged as invitations to build number sense; some of these are amazingly helpful, while others are busy work that don't necessarily invite mathematical thinking. Before going to the trouble to reproduce a game and organizing all the required pieces, ask yourself, "In what ways will this game build or sustain mathematical understanding?" If that is unclear, consider modifying the game to require more explanation, justification, or representation—or perhaps set it aside. Time is precious in math class; let us devote as many minutes as we can to tasks that truly deepen learners' understanding.

For ideas on number talks and number sense routines, you may explore the excellent resources by author-educators Sherry Parrish (2022), Jessica Shumway (2011), and others to identify regular practices that serve learners. To be worthwhile for all students, these activities and conversations need to be facilitated in a consistently engaging and inclusive manner, with all learners expected and accountable for diving into the thinking and sharing their mathematical ideas. Make sure there are no hiding places for students in your numeracy routines.

Reflection
What is the status of your students' number sense?

Number sense is a sort of mathematical calisthenics designed to warm up and stretch the mind in the same ways that jumping jacks might warm us up to take the field for a ball game. These skills of understanding bare numbers and their relationships are critical to the development of numeracy and yet are not an end in themselves; number sense prepares learners for more complex, applied challenges as problem solvers. Students with number sense can step back from a solution and notice whether it seems accurate and makes sense, given the context and the mathematical operations at play.

Conceptual Understanding

Historically, American math education has focused extensively on skill building and number sense, plugging and chugging numbers, finding answers to brief questions—and quickly. International comparative assessments have demonstrated our students' deficits solving complex, novel

problems. To do so requires conceptual understanding, which takes time to develop. Educators rushing to cover standards to stay on pace may feel tempted to shy away from understanding and focus instead on procedures—but at students' peril. Understanding allows us to remember and reapply concepts in unfamiliar situations.

Conceptual understanding is the antidote for rote procedures, many of which I know I learned as a young mathematician: move a decimal point a bunch of places, cross multiply and divide to multiply fractions, FOIL an equation, and more. At the time, I didn't understand why those tricks worked or how they might be applied in real-world scenarios. This made me robotic in my execution of math procedures and confused when the circumstances changed: *Wait, what? There is a decimal inside a fraction? A fractional numerator? Three binomials to multiply instead of two?* Without understanding the meaning of the methods I had rehearsed, I was sunk. But, I asked questions and clawed my way to understanding.

An instructional focus on conceptual understanding allows students to remember a mathematical idea and to apply it flexibly, as needed, with confidence and precision. Our quest to scaffold our learners' understanding is an ongoing, up-growing spiral with stages of progress along the route. When a student truly understands, according to *Understanding by Design* authors Grant Wiggins and Jay McTighe (1998), they can explain, interpret, apply, provide perspective, demonstrate empathy, and reflect on their self-knowledge.

If a student strongly understands subtraction, for example, they are able to not only solve a naked number problem but also to come up with more than one method to determine how many seats will be empty after a group of thirty-eight has climbed on board a bus that has thirteen rows of four seats each. They will be able to represent their calculation in a variety of ways, though ultimately, there will be one and only one correct numeric solution. They can explain their solution in terms of addition and subtraction, compare it to the thinking of others, and justify and confirm its accuracy. Once their interest is sparked, they are also more likely to think about buses and ridership differently next time they climb aboard. The math concept's connection to their experiences makes it far more relevant and interesting than simple numerical problems on a worksheet.

When we comprehend mathematical concepts, we can generalize from a specific case to a set of principles or patterns, and vice versa, in ways

that increase efficiency and foster discussion. We leverage our understanding of prior concepts—say, multiplication with whole numbers—to build our understanding of more advanced concepts such as multiplication of polynomials. Understanding builds the foundation for future learning and lifelong mathematical success. Math workshops offer us the time and space to delve into the important work of making meaning as mathematicians.

Reflection

What is a mathematical concept you understand deeply? How did you arrive at that understanding? How many ways can you represent that idea?

Modeling

A fruitful pathway toward understanding is the use of models to represent concepts. In recent years, math curricula have emerged with all manner of models, from manipulative plastic cubes to tape diagrams to fraction strips to graphed functions on a device. Some educators have done away with three-dimensional scaffolds and focus on the abstraction of screen-based visuals, including graphic calculators or other online animations. These resources are all tools for learners to generate their own conceptual understanding. It's critical to remember, though, that these models represent possibilities, not necessarily the one and only one way to "see" a mathematical idea. Rather than focusing solely on their mastery of a tree diagram, for example, students can be well served when we present more than one model of a concept and invite them to generate their own. We need not teach every possible model any more than we ought to obsess over one. The model is not the concept—but merely one representation of that concept—so let us hold it loosely, as a tool, with the mathematical idea itself as the end goal for mastery.

Conceptualizing Mathematical Ideas

Conceptual understanding is not necessarily passively constructed while lecture-listening or video-watching unless learners are particularly curious or intentionally engaged. Understanding is an active process, requiring higher-order thinking across time. To internalize an understanding, brain research tells us, we must work with a concept ourselves. This process of connecting learners' minds with ideas and vice versa can be achieved in a variety of ways, including through modeling, open-ended tasks, and conversations.

When preparing to present a new idea to students, consider their unique background knowledge and cultural contexts and how to invoke those as starting points, then imagine what sorts of experience might

scaffold their progress toward an *"aha"* moment. Is there a familiar data set they might explore—about heads of livestock, if you're teaching in a ranching community or about tides, if you live near the coast? Is there a pattern they could discern—of, say, where or when, buses stop along certain routes in your city? A solution pathway they could correct—a common error? An area in their lives where the math idea you're teaching is essential? These questions can help us to craft thoughtful challenges and associated minilessons, as will be discussed in greater detail in Chapters 5 and 6.

Time

To bother with understanding takes time and effort. Taking time for students to marinate in mathematical ideas is an investment in long-term prowess that pays off in their ability to transfer their understanding to novel contexts. According to researcher Manu Kapur (2024), author of *Productive Failure: Unlocking Deeper Learning Through the Science of Failing*, the experience of confusion—even failure—as we grapple to make meaning is an essential step toward comprehension. The disequilibrium of confusion creates the desire to understand and ensures that a concept, then learned, will be remembered and flexibly reapplied in future contexts.

While the mounting pressures on teachers today compel us to feel rushed, we know that students remember topics that we take time to study in depth and with relish. Resist the temptation to be satisfied with mere box-checking and instead leverage your time to favor in-depth meaning making of essential concepts that students can carry with them throughout their lives.

> Teacher Jessica Piwko, for example, set aside a whole lesson so students could learn the Pythagorean theorem for themselves by analyzing data rather than being lectured on the rule about the relationship between right triangles' side lengths. "What patterns do you see?" she asked as students leaned over a data set they created listing the side lengths of some right triangles, after measuring them with metric rulers. Rather than tossing learners an age-old formula, Ms. Piwko committed a day of hands-on, minds-on learning so that students could discover for themselves the beloved math rule about

the relationship between right triangles' side lengths—and so they will understand it more deeply and remember.

"Side c is always longer," one student offered.
"Although you can switch side a and side b, c was always greater," another added.
"If you square a and b, add them, they make the square of c," a third teammate chimed in.
"Does it work every time?" Ms. Piwko asks.

The students at this table agree that it does.

"Who sees that now?" she asks the group. Many nod. "Tomorrow, we will put this into an actual formula and write that pattern out."

Problem Solving

Problem solving is the fourth aspect of being a capable mathematician—and, essentially, the point of doing math in the first place. When we, as math learners, have can-do attitudes, when we understand the nature of the numbers at hand and the operations or concepts we are being asked to apply to them, we can effectively solve problems. University of Delaware mathematics education professor James Hiebert described the importance of problem solving for our learners in his book *Making Sense,*

> *Today's students need flexible approaches to problem solving methods that can be adapted to new situations, and they need know-how to develop new methods for new kinds of problems. Nowhere are such approaches more critical than in the mathematics classroom.* (Hiebert et al. 1997, 1)

While solving problems in math is certainly a worthy calling and one with important life applications ranging from paying bills to calculating taxes, life is full of seemingly un-mathematical problems that also call for solutions: We lose our keys, get a flat tire, clog the toilet. While solutions

to some of these adult foibles don't necessarily require math, they do call upon us to use some of the same skills—productive dispositions and conceptual understanding to solve. We can remind learners that the qualities and insights they are developing as mathematical problem solvers will serve them well throughout their lives. To emphasize this, we can tell stories of even the most mundane problems we solve in our lives throughout the days and weeks of the school year. Sharing our own challenges not only humanizes teachers and adults, but also demonstrates the reality that life is all about figuring things out and that we—like our students—are "figure it out" sort of folks, as educator Peter Johnston (2012) aptly described in his book *Opening Minds*.

While teaching math, we are building habits of mind. As true problem solvers, capable mathematicians are able not only to check the right box on a multiple-choice test, but also to persevere and prevail through mathematical and real-life challenges.

Practices

The authors of the Common Core standards present a set of experiences that mathematicians at all levels can engage in regularly, regardless of which math topic they may be exploring. The Standards for Mathematical Practice invite students to:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning. (Common Core State Standards Initiative n.d., 6–8)

All eight practices can be facilitated handily within the context of math workshops, as will be described in greater detail in Chapter 3. For now, let us appreciate that problem solving is complex, flexible, and involves a broad range of strategies. We can foster the development of learners' problem-solving prowess when, for each lesson, we intentionally align a concept or task with a focus on one of the eight standards—a process learning target—essentially feeding two birds with one piece of bread. We can practice attending to precision while multiplying decimals, for example, or rehearse critiquing the reasoning of others while analyzing data. In this way, the specific content target of the day can be explored while students also practice a broader, transferable math process that supports problem solving now and in their futures. When we are transparent about these invitations and overlaps, we build students' self-awareness of their own growing cognitive toolkits.

Grappling

Researcher Manu Kapur, who studies the role of struggle in mathematical progress, identified three conditions, all helpful, though tricky to implement:

> *"First, choose problems to work on that 'challenge, but don't frustrate.' Second, provide learners with opportunities to explain and elaborate on what they're doing. Third, give learners the chance to compare and contrast good and bad solutions to the problem."* (quoted in Paul 2014)

While teachers and students alike love answers, Kapur's research informs us that a great deal of progress toward understanding can be made in the murky waters before an accurate solution is settled. For this, we can slow learners down in their rush for answers and welcome them to notice and name their progress as grapplers:

- What are you doing here?
- Why are you doing that?
- What are you thinking now?
- How will you proceed from here?

Math workshop provides us with the time and structure to have these conversations both with individuals and with our whole class. When we teach learners to honor their own process—the mental gymnastics of mathematics rather than simply the results—we can encourage more and more minds to engage in mathematical work.

Teacher Erica Vannest encourages such conversations with a problem on compound interest. Instead of urging students to jump in and solve it, she slows them down, distributes highlighters, and asks them to identify what they understand and where they have questions as they read through the scenario. After a few minutes of quiet reading, Ms. Vannest pairs learners up and invites them to discuss the text with peers using supportive sentence stems on the slide:

"I understand . . ."
"I am unsure about . . ."

Reflection
In what ways do you facilitate learners' perseverance as problem solvers?

After these conversations, Ms. Vannest underscores the value of thinking and talking a problem through with peers before seeking an answer, celebrates her students for their effort, and encourages them to use what they learned in these brief conversations to get started.

"You got this!" she calls out to the class.

They dive in.

Purposeful Planning

To cultivate capable mathematicians with these four strengths—productive dispositions, number sense, conceptual understanding, and problem solving—we need to be planful. It is not enough to just keep these ideas in mind or hang posters with these terms on our walls. We need to intentionally design for integration of these goals within our unit and lesson planning, as well as narrate conversations about these targets with learners often.

As you design each math unit, consider what opportunities you have to weave in the four strengths of capable mathematicians across time. You might ask yourself:

- Productive Dispositions: How might this experience build learners' identities as mathematicians?
- Number Sense: What aspects of number sense can I build within this unit?
- Conceptual Understanding: How will I facilitate mathematical meaning making?
- Problem Solving: In what way will I support learners' process as problem solvers?

Mindful of the big picture, we can leverage daily math learning toward our highest hopes for students. More ideas on how will be included in later chapters.

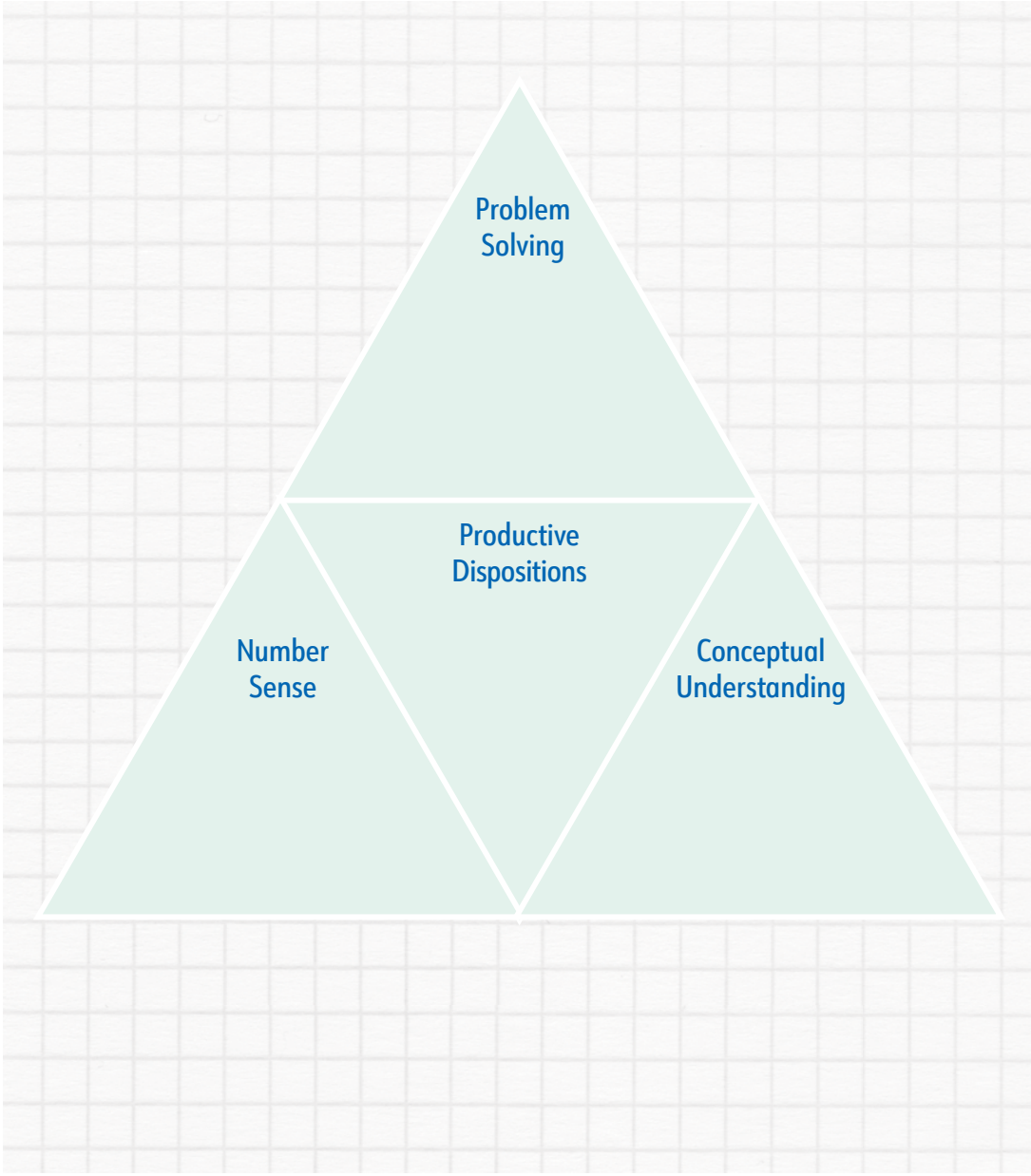


Figure 1.2 Planning for Capable Mathematicians

(Adapted from Kilpatrick, Swafford, and Findell 2001)

Solution: Know Why

The stakes are high. Our students need us to be on purpose in our work every day. With intention and planning, we can leverage the abundant lists of math content standards to address and curricula to cover in service of our good intentions. Students can use their learning time in math every day to develop the productive mathematical dispositions, number sense, conceptual understanding and problem-solving skills they need to prevail as capable mathematicians. Let us begin with this end in mind.

Now that we have oriented to the why of our work, Chapter 2 will explore our stances, how we as adults can effectively lead math learners. Then, the balance of the book will provide more specific insights into the design and facilitation of daily math workshops that connect our greatest hopes with skillfully designed and facilitated opportunities for learners to shine.

Synthesis Questions

- How has your own experience with math impacted your life?
- How does this chapter's vision for capable mathematicians square with your own?
- What do you see as the relationships between productive dispositions, number sense, conceptual understanding, and problem solving?
- What are your learners' strengths and areas of growth as capable mathematicians?
- As a result of your learning in this chapter, what is something new that you will implement? How?