RESEARCH EVIDENCE BASE

math expressions

Heinemann

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Introduction

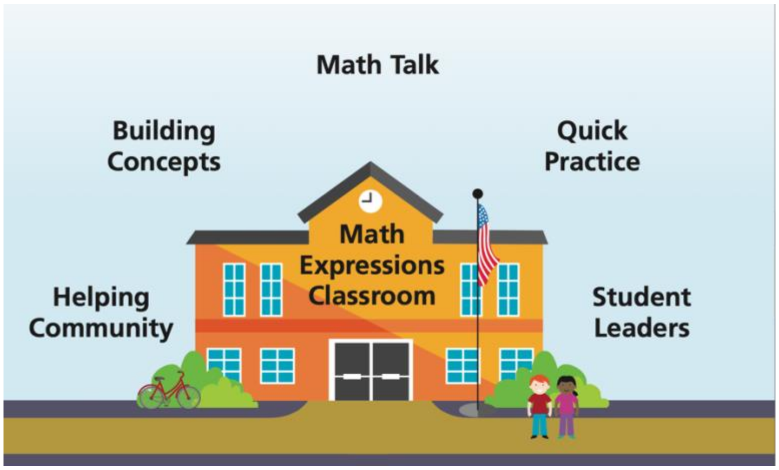
Math Expressions © 2026 is a research-based, NSF-funded, comprehensive, coherent, cumulative, rigorous, balanced, mathematics program for Grades K–6, authored by Dr. Karen Fuson, a leader in the field of mathematics education. Through proven approaches, *Math Expressions* develops deep mathematical understanding and fluency and yields demonstrated achievement for all students within an inquiry-based curriculum.

Math Expressions © 2026 helps children make sense of math by exploring and explaining their knowledge of key concepts, via numbers, words, and visuals. At the heart of *Math Expressions* is the cultivation of a **Math Talk Learning Community**. Through their experiences in this rich, collaborative environment, students reach their learning destination—the ability to use formal math methods with understanding and fluency.

Mastering math starts with knowing the numbers. While *Math Expressions* © 2026 provides students with a solid foundation in numeracy, the program also aids students in developing broader understanding of how and why numbers work, encouraging students to ask the questions and apply their ideas in ways that lead to a thoughtful, informed, approach to solving problems and mathematical habits of mind. In highly interactive, engaging lessons within a **Class Learning Path** framework that affords mastery for all, students learn how to think more deeply and choose their own strategies and solutions to the answers—experiences that will take them far beyond the math classroom.

When children understand math, they're not relying on memorization; they're relying on themselves. Dr. Karen Fuson's comprehensively researched approach to teaching math is based on the way children actually learn. The program has been used across the country and the world for decades because it raises student achievement and provides learners with rich experiences early on to support their success in life, whether their path takes them into STEM careers or engaged citizenship within any community.

The *Math Expressions* program centers around five core classroom structures: **Building Concepts**, **Math Talk**, **Student Leaders**; **Quick Practice**, and **Helping Community**. These structures are not static. They are interactive. They work together and support one another. Taken together, the five core structures help ensure that children from all backgrounds, with diverse learning and life experiences, learn mathematics at a deeper level with understanding, fluency, and confidence.



Within the *Math Expressions* curriculum, a series of learning progressions reflect research on students' natural learning stages when mastering concepts such as computation and problem solving strategies. These learning stages informed the order of concepts, the sequence of units, and the positioning of topics across the program. Throughout *Math Expressions*, teachers guide students over four teaching phases and lead them along their own inquiry learning paths toward mathematically desirable outcomes:

- **Guided Introduction** brings out students' prior knowledge.
- **Learning Unfolds** as students form conceptual networks and use mathematically sound methods.
- **Knead Knowledge through Practice** to ensure that students gain understanding and fluency with desired methods.
- **Maintain Fluency** as students remember and relate new topics to prior knowledge.

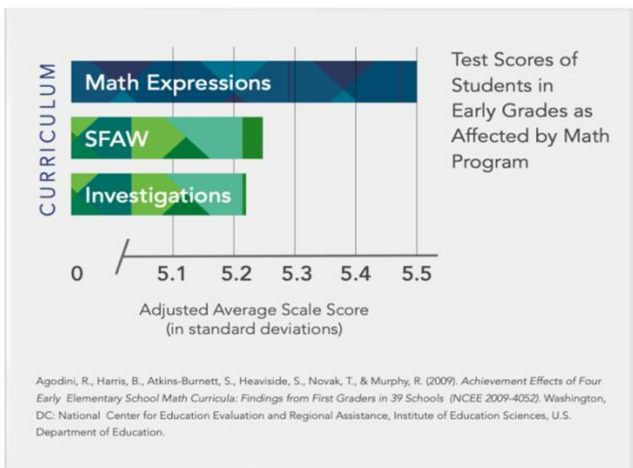
Integrated with *Math Expressions* © 2026 is the award-winning, game-based, digital, and adaptive program ***Matific***. This combined solution enhances learning by empowering students to explore math with creativity, engagement, independence, and community along a personalized learning pathway. Available on the Heinemann *Flight* platform, *Matific* provides each student with curated, customized practice in key mathematics skills and topics aligned to the *Math Expressions* curriculum as well as rigorous standards.

matific

Key Research in the Development of Math Expressions ©2026

Math Expressions was developed using the methods of learning science design research focused on building students' conceptual understanding of math, interwoven with the other components of math proficiency.

The program was part of a large-scale study sponsored by the U.S. Department of Education to determine the effectiveness of four elementary math curricula on student academic achievement. The results of the study showed that students using Math Expressions were performing at higher levels of math achievement when compared with students using similar programs. Math Expressions yielded substantial increases in test scores as well as on broader measures of understanding.



The Author—Dr. Karen Fuson, Professor Emerita of Education and Psychology at Northwestern University and author of *Math Expressions*, is a mathematics educator and developmental and cognitive scientist with decades of experience studying, researching, and writing about mathematics instruction.

Dr. Fuson's research focuses on how children learn math and the classroom conditions that support the development of students' understanding. Dr. Fuson's research for her *Children's Math Worlds* (CMW) NSF-funded project was instrumental in identifying key components for successful mathematics learning: **Building Concepts, Math Talk, Student Leaders, Quick Practice,** and **Helping Community**. The body of research that forms the basis of Math Expressions focused on the following research tasks:

- Analyzing real-world mathematical situations to help curriculum developers and teachers select problems and examples that ensure both the understanding of the general math principles at work and of the real-world situation itself.

- Analyzing formal mathematical language and notation to identify difficulties that need to be addressed with pedagogical supports and classroom discussion.
- Developing meaningful real-world situations and visual supports that can facilitate interest and accessibility.
- Identifying meaningful language that can connect to the formal mathematical language (e.g., “break-apart partners” for addends, “unmultiplying” for dividing).
- Identifying typical student solution methods and learning paths through a domain to more-advanced solution methods.
- Developing accessible but mathematically desirable algorithms that relate to common algorithms but that all students can understand and explain.
- Identifying typical student errors and how to overcome them.
- Choosing drawn quantity representation that can facilitate understanding of the domain situations or quantities.
- Monitoring grade-level placement of, and approaches to, important topics around the world.
- Writing teaching materials in a “learn while teaching” style that enables teachers to learn new ways of teaching and new solution methods.
- Developing classroom activity structures that can be used repeatedly with different math topics to cut down on classroom management issues.

From her research results and collaborations with and knowledge of the research of others in the field, Dr. Fuson has designed effective teaching approaches and identified progressions of Pre-K to intermediate-level students’ development/experiential understanding across different mathematical domains. In addition to the research projects described within this document, Dr. Fuson served on the National Research Council committees (below) that summarized research and made recommendations.

Math Expressions program is the result of this extensive research into how students learn math.

Key Research—As evidenced by the numerous studies and program references presented throughout this report, *Math Expressions* reflects what research shows about effective mathematics teaching and learning. The following publications, cited in full at the end of this document, include the work of Dr. Fuson and present key findings foundational to the development of *Math Expressions*.

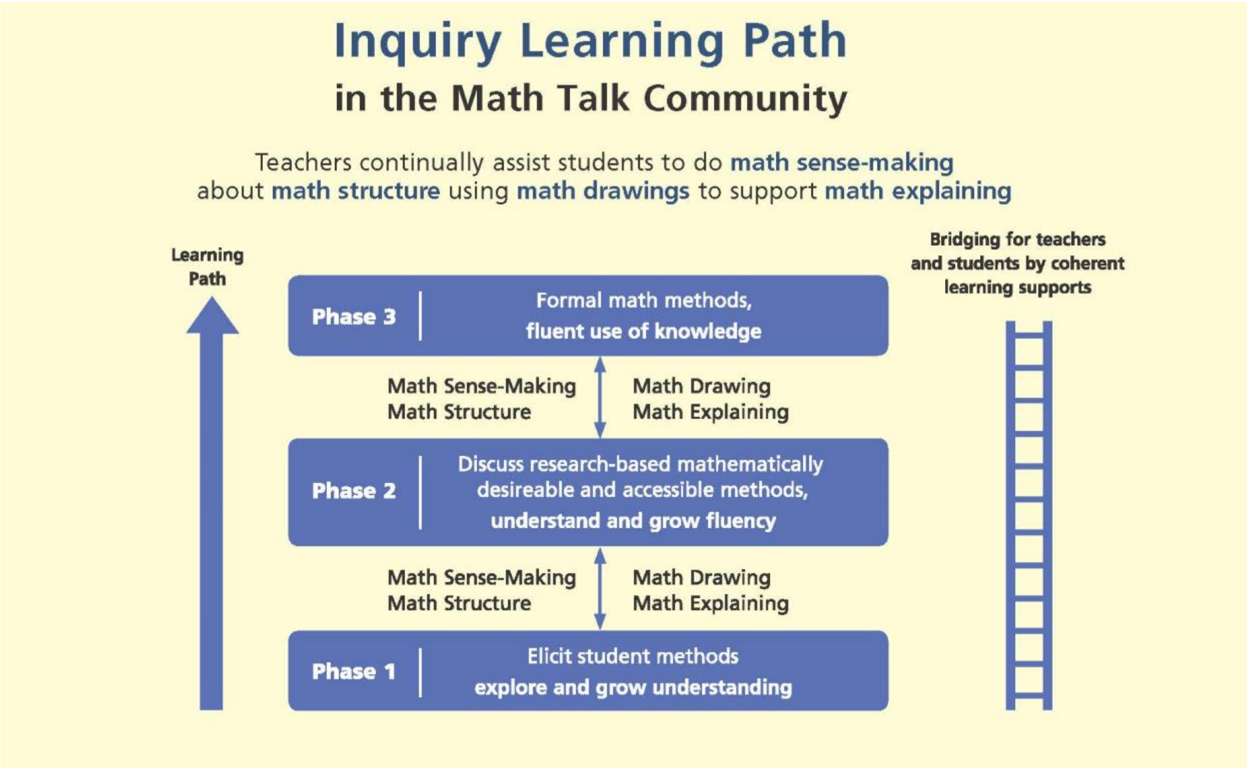
> *Adding It Up: Helping Children Learn Mathematics* (National Research Council, 2001). *Adding It Up* presents a picture of mathematics learning from Pre-K to Grade 8. The Mathematics Learning Study Committee identified five components of mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition) and presents research findings for how students develop this proficiency. The book provides key recommendations for specific changes and approaches in teaching, curricula, and teacher education that can improve students’ mathematics learning.

How Students Learn: Mathematics in the Classroom (National Research Council, 2005). This publication was written to build upon the earlier How People Learn, and to tailor the findings in a more practical, useful way that teachers can immediately employ in their instructional practices. Full of detailed suggestions for research-based instructional activities, the book is designed to help teachers meet challenges and develop understanding, fluency, and problem solving among their students.

Mathematics Learning in Early Childhood: Paths toward Excellence and Equity (National Research Council, 2009). The result of a comprehensive review of the research on mathematics learning in early childhood, this publication identified critical areas for early mathematics study that will enable all students to reach their potential in mathematics. The research reported suggests that improvements in early childhood mathematics education will also particularly support those students at risk of falling behind in mathematics by providing them the strong foundations they need for future success.

Math Talk—The decade-long, National Science Foundation-funded Children's Math Worlds (CMW) Research Project that developed the curriculum on which Math Expressions is based found that using Math Talk structures in the classroom enables children from all backgrounds to learn mathematics at high levels of rigor and with understanding, fluency, and confidence.

Class Learning Path Model—The Class Learning Path model for teaching mathematics for understanding and fluency that is central to *Math Expressions* is rooted in decades of research conducted by program author Karen Fuson and other leaders in the field, both in the U.S. and internationally. Through its years of research and development, the model is discussed in detail in multiple publications cited at the end of this document, including Murata & Fuson, 2006; Fuson & Murata, 2007; Fuson, Murata, & Abrahamson, 2014; Murata, Fuson, & Abrahamson, 2020. The Class Learning Path model integrates two theoretical frameworks—a Piagetian focus on learning and a Vygotskiian focus on teaching. As described in Murata et al. (2020): "This "balanced teaching" approach is related to several models of teaching that emphasize learning trajectories (Clements & Sarama 2004, 2014), task sequences (Simon, Placa, & Avitzur, 2016), "math talk" in the classroom (Yackel & Cobb 1996; Hufferd-Ackles, Fuson, & Sherin 2004, 2015; Murata, et al. 2017), linking ideas (Alibali et al. 2013), embodied thinking (Abrahamson, 2014), visual learning (Mason, 1989), productive failure (Kapur, 2014), and discovery-based learning (Abrahamson & Kapur, 2018)" (p. 62). In 2009, Fuson discussed extreme views of teaching that are sometimes drawn from a Piagetian view ('learning without teaching' that values only children's invented methods) and from a Vygotskian view ('teaching without learning' with a traditional emphasis on fluency, rather than on meaning making) and presented a balanced learning–teaching approach relating these perspectives. In this approach, called learning path teaching, mathematically desirable methods that are accessible to children, and may have been invented by children, are linked to and explained using maths drawings or other visual referents to support meaning making. This approach enables all children to use general methods with understanding and move to fluency. Major aspects of the model were drawn from principles in two National Research Council reports on research on math teaching and learning (Donovan & Bransford, 2005; Kilpatrick, Swafford, & Findell, 2001) and from the National Council of Teachers of Mathematics process standards (NCTM, 2000) in the United States. The classroom instructional conversations support individualized instruction within whole-class activity as the methods of all students appear and are discussed. Diversity can be accepted and used to increase understanding by all, but the Class Learning Path model also assumes and makes possible the realization of high academic expectations by the early introduction and support of visual models and methods in the middle (Murata, 2013).



Math Talk Community

Math Talk encourages students to actively, strategically, meaningfully participate in their learning and provides students the opportunity to learn from each other, yielding higher levels of skill, engagement, and confidence for students and teachers alike (Clements, Lizcano, Sarama, 2023; Hufferd-Ackles, Fuson, & Sherin, 2004 & 2015; Humphreys & Parker, 2015; Michaels, O'Connor, Hall, & Resnick, 2010; Morgan, Craig, Schütte, &, Wagner, 2014; Parrish & Dominick, 2016; Saylor & Walton, 2018; Wagganer, 2015). The process of having students talk about their mathematical thinking makes learning visible, providing teachers with a means of addressing students' misconceptions and moving students from surface to deeper understanding (Fuson, Kalchman, & Bransford, 2005; Fuson & Leinwand; Hattie & Clarke, 2018; Hattie, Fisher, & Frey, 2017). Research firmly supports Math Talk as an equitable teaching process that benefits students at different levels of learning and in different contexts, including English learners and historically marginalized students (Bray, Dixon, & Martinez, 2006; Fuson, Atler, Roedel, & Zaccariello, 2009; Fuson, De La Cruz, Smith, Lo Cicero, Hudson, Ron, & Steeby, 2000; Fuson & Leinwand, 2023; Murphey, Madill, Guzman, 2017; Turner, Dominguez, Maldonado, & Empson, 2013).

A classroom in which meaningful communication and discussion are primary vehicles for learning and in which members co-construct and support one another's understanding is known as a "Math Talk Learning Community;" within effective math talk communities, teachers shift from the traditional role of directing all learning to one of a coach or facilitator who promotes greater student agency and ownership in a student-centered learning environment (Hufferd-Ackles, et al., 2004 & 2015; Saylor & Walton, 2018; Thanheiser & Melhuish, 2023). The frequency of teachers' math talk has been shown to correlate with students' increased mathematical knowledge (Klibanoff, Levine, Huttenlocher, Vasilyeva, & Hedges, 2006; von Spreckelsen, Dove, Coolen, Mills, Dowker, Sylva, Ansari, Merkley, Murphy, & Scerif, 2019). Math Talk conversations act as scaffolds for students developing mathematical language because they provide opportunities to simultaneously make meaning and communicate that meaning (Mercer & Howe, 2012; Thanheiser & Melhuish, 2023; Zwiers, 2014). Nurturing Math Talk Learning Communities are developed deliberately over time and in the process positively transform learning experiences and environments—both cognitively and affectively—for students and teachers (Fuson & Leinwand, 2023; Fuson, Murata, & Abramson, 2014; Parrish, 2010; Parrish & Dominick, 2016; Yackel & Cobb 1996). "The informal and formal representations and experiences need to be continually connected in a nurturing 'math talk' learning community, which provides opportunities for all children to talk about their mathematical thinking and produce and improve their use of mathematical and ordinary language" (Cross, Woods, & Schweingruber, 2009, p. 43).

Math Expressions ©2026's longstanding vision for Math Talk is an instructional conversation within a nurturing, supportive classroom environment guided by the teacher but with as much student-to-student talk as possible and focused on developing the understanding of all students. Within *Math Expressions*, Math Talk practices are integral to helping each and every student along the learning path for a given mathematical topic—and the program fosters the establishment and growth of positive, productive Math Talk Learning Communities.

What Is a Math Talk Learning Community?

It is well established in research and practice that having students communicate mathematically and engage in discourse about their math learning is an essential, core component of effective mathematics education. This has been recognized by the National Council of Teachers of Mathematics (2000, 2014) and the National Research Council (2001, 2004). Mathematical communication "can support students' learning of new mathematical concepts as they act out a situation, draw, use objects, give verbal accounts and explanations, use diagrams, write, and use mathematical symbols...the communication process also helps build meaning and permanence for ideas and makes them public" (NCTM, 2000, p. 59-60). A major factor impacting teaching and learning of mathematics is the quality of classroom discourse (Hiebert & Wearne, 1993; Smith & Stein, 2011/2018).

Mathematical discourse—speaking, writing, or listening about mathematics—is an important way for students to learn and make sense of mathematics; such communicative exchanges provide access to ideas, relationships among those ideas, strategies, procedures, facts, and mathematical history as well as foster deeper understanding and positive attitudes toward mathematics (Chapin, O'Connor, & Anderson, 2009; Humphreys & Parker, 2015; Leinwand & Fleischman, 2004; Michaels et al., 2008; Morgan, et al., 2014; Smith & Stein, 2011/2018). Discourse within mathematics learning settings, especially when marked by teachers' encouragement that students verbalize their thinking and understanding and their provision of feedback to students on that shared verbalization, has been shown to benefit students across grade levels in their development of reasoning and problem-solving skills (Humphreys & Parker, 2015). "Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and argument" (NCTM, 2014, p. 29).

For decades, Math Talk has been identified as an essential component of mathematical thinking (Cuoco, Goldenberg, & Mark, 1996) as well as an equitable practice (Fuson & Leinwand, 2023). Math Talk has a well-established research base. Inagaki, Hatano, & Morita (1998) found that students who discussed and justified their solutions with peers demonstrated greater mathematical understanding than students who did not engage in such discussions. Instructional practices of restating, prompting students, and engaging in whole-class and small-group discussions, and paired conversations have been shown to be effective in improving student understanding (Chapin, O'Connor, & Anderson, 2003). Leinwand and Fleischman's 2004 research review on effective math instruction reported that talking about math and explaining the rationale for solutions builds conceptual understanding. In a study of how teachers' language use impacted students' mathematical knowledge, Klibanoff et al. (2006) concluded that that the frequency of teachers' math talk correlated with students' increased mathematical knowledge. In 2019, von Spreckels and colleagues found that when children were able to participate in classroom-based math discussions, their understanding of complex mathematical concepts increases and outcomes related to cardinality skills were greater in settings where teachers utilized a wider breadth of mathematical language.

Talking about math has been demonstrated to positively impact underrepresented students and English learners at varying levels within math classrooms (Fuson et al., 2000; Hufferd-Ackles et al., 2004 & 2015; Murphey et al., 2017). As students in transitional language classrooms engage in Math Talk, they “communicate verbally and in writing about their mathematical ideas; they not only reflect on and clarify those ideas but also begin to become a community of learners” (Bray et al., 2006, p. 138). Math Talk with teachers and peers that includes prompt feedback is also beneficial for students with mathematical learning disabilities (Clements & Sarama, 2012). Other researchers such as Mercer & Howe (2012), Zwiers (2014), and Cross et al. (2009) have noted that because it requires students to concurrently make and communicate meaning, Math Talk provides scaffolding for students to build mathematical language. "To help students become mathematically proficient, reason mathematically, and compute with accuracy, flexibility, and efficiency, teachers must provide opportunities for students to grapple with constructing number relationships, test these relationships, and analyze and discuss their reasoning" (Parrish & Dominick, 2016, p. 15).

Math Talk also provides teachers with rich assessment that elicits insight about student thinking and how responses are arrived at, which can lead to even more impactful discussions about conceptions and misconceptions (Fuson, et al., 2005; Fuson & Leinwand, 2023; Hattie, et al., 2017).

"Mathematical conversations provide opportunities for teachers to hear regularly from their students and to learn about the range of ideas students have about a particular mathematical idea, the details supporting students’ ideas, the values students attach to those ideas, and the language students use to express those ideas. The knowledge teachers gain from engaging with their students in conversations is essential for teaching for understanding” (Franke, Kazemi, & Battey, 2007, p. 237).

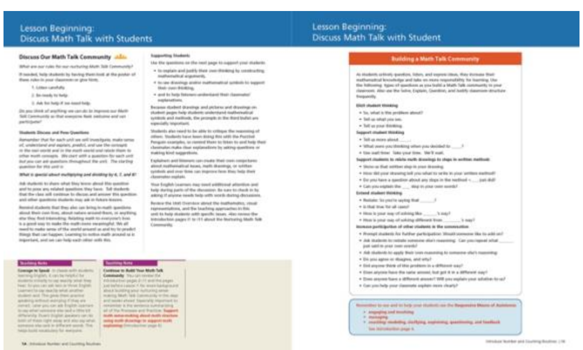
Classroom discussions can be organized in ways that have been shown to support the acquisition of mathematics concepts and language development. “The teacher’s role in discussions is critical. Without expert guidance, discussions in mathematics classrooms can easily devolve into the teacher taking over the lesson and providing a “lecture” on the one hand or, on the other, the students presenting an unconnected series of show-and-tell demonstrations" (Smith & Stein, 2011/2018, p.4). Results of the Children’s Math Worlds (CMW) Research Project that developed the curriculum on which Math Expressions is based found that using Math Talk structures in the classroom enables children from all backgrounds to learn mathematics at high levels of rigor and with understanding, fluency, and confidence (Fuson, 1998; Fuson, et al., 2009; Fuson, et al., 2000; Hufferd-Ackles et al., 2004).

In a Math Talk Learning Community, “individuals assist one another’s learning of mathematics by engaging in meaningful mathematics discourse” through a process that relies on students' problem solving and reasoning skills, which then also yields deeper learning (Hufferd-Ackles et. al, 2004, p. 81). Math Talk Learning Communities are further marked by their student-centered instruction and classroom environment, with teachers in the role of coach or facilitator (Fuson et al., 2005; Hufferd-Ackles et al., 2004 & 2015; Humphreys & Parker, 2015; Parrish & Dominick, 2016; Saylor & Walton, 2018) Teachers guiding collaborative conversations focused on the mathematical problems and thinking, with both the teacher and students utilizing responsive means, via engagement, modelling, clarifying, explaining, questioning, and feedback, to promote and support the full participation of the class, including in its management and coaching roles (Fuson, et al., 2014). A Math Talk Learning Community is intentionally developed over time, to create a classroom culture that is cohesive, inclusive, and safe for all, rooted in mutual respect and common goals and interests in learning (Fuson & Leinwand, 2023; Parrish, 2010; Parrish & Dominick, 2016).

How *Math Expressions* Delivers

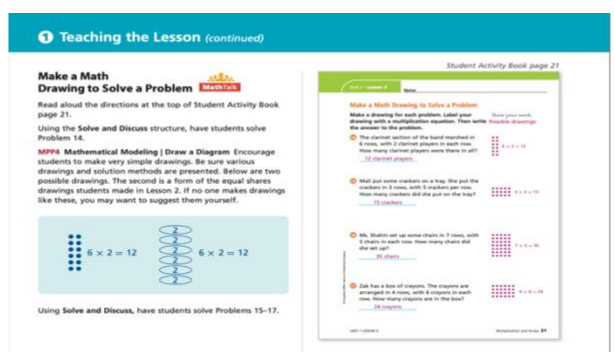
Math Talk—and the **Math Talk Community** to nurture and facilitate this research-proven practice—has been at the heart of the *Math Expressions* program for decades and key to its demonstrated efficacy. Math Talk classrooms fostered by *Math Expressions* create lively, collaborative, inquiry-based, and well-structured learning environments where children share their ideas and solutions and listen respectfully to the ideas of their classmates. Through their experiences in this rich Math Talk community, students reach their learning destination: the ability to use formal math methods with understanding and fluency.

Math Talk as supported by *Math Expressions* generates constructive discussion of problem-solving methods through well-defined classroom activity structures comprised of four components: **questioning, explaining math thinking, contributing math ideas**, and **taking responsibility for learning**.



For all major, grade-level topics, *Math Expressions* starts at each student's level and continually elicits their thinking, provides visual and linguistic supports to move them to understanding, and ends with extended fluency practice, while continuing the emphasis on understanding and explaining with Math Talk. Each lesson includes a complete description of the activity and provides explicit guidance for what teachers should expect from students, as well as explanations, sample questions, and student/teacher dialogs for Math Talk.

Math Talk offers ongoing opportunity for students to apply and extend their mathematical thinking, drawing, questioning, justifying, and problem solving throughout learning activities. *Math Expressions* uses worked examples in the classroom by having students solve and explain methods to their classmates during the frequent Math Talk parts of a lesson. Math Talk opportunities are noted to teachers throughout the **Teacher Editions** and accompanied by ongoing, point-of-use support in how to effectively guide Math Talks within a nurturing learning community.



Building a Math Talk Community

To promote a Math Talk Community, teachers must pivot away from traditional modes of instruction that put themselves at the center, directing and driving all learning activity; instead teachers in effective Math Talk Communities act as student-focused coaches or facilitators who listen, question, and encourage increased student agency and frequent peer-to-peer conversations about math and math learning (Fuson et al., 2005; Hufferd-Ackles et al., 2004 & 2015; Humphreys & Parker, 2015; Parrish & Dominick, 2016; Saylor & Walton, 2018). In Hufferd-Ackles and colleagues' research (2004, 2015), they articulated a framework found to yield success in an urban neighborhood in a class with English learners. They found that creation of a math-talk community requires teachers to shift from the level of a traditional, teacher-directed classroom (Level 0 in their framework) to a classroom in which teachers coach and assist as students take leading roles (Level 3). To do so, students need to develop skills across the components of questioning, explaining mathematical thinking, identifying the source of mathematical ideas, taking responsibility for learning, and mathematical representations (Fuson, et al., 2014; Murata et al., 2020).

Fuson and Leinwand (2023) distinguish between a "number talk" and an authentic Math Talk classroom. They point out that a number talk is an instructional strategy, usually occurring at the start of a math lesson, that engages students in problem- or topic-based discussion of the potential approaches, misconceptions, and errors and serves to encourage agency and deeper thinking. Number talks are an ongoing element within a Math Talk Learning Community, which is a deliberately, gradually developed, nurturing and supportive learning culture and environment with a consistent focus on student discourse across core mathematics instruction. Within Math Talk Communities learning relies on students' written and spoken expressions of multiple representations and meets the criteria of Level 3 described with the framework established by Hufferd-Ackles and colleagues (2004, 2015).

Effective Math Talks encourage students to approach and solve problems in multiple ways. This includes mentally, helping students to focus on number relationship and strategy usage to construct mathematical ideas. In ensuing conversations and discussions, students should share, clarify, and defend their solutions and strategies and collectively reason about numbers and concurrently building understanding (Parrish & Dominick, 2016). A key Math Talk Community activity has students with incorrect answers revise their thinking and understanding and to share with other students their justifications and rationales as well as their thought processes leading to correct answers, so that the larger group benefits (Humphreys & Parker, 2015; Parrish, 2010). Teachers must also provide conceptual support for the thinking and problem-solving students do while engaged in Math Talk and math drawings are effective for this; including visuals within a Math Talk Learning Community context can be particularly supportive of English learners and historically underserved students (Fuson et al., 2005; Fuson et al., 2009).

Strategies that effective Math Talk Community teacher-facilitators use include discussing why math and math talk is important, providing examples, introducing sentence stems, modeling how to listen and respond, and offering contrasting explanations instead of justifications (Wagganer, 2015). It is critical that in the course of Math Talk the students themselves share, evaluate, and reflect upon their own strategy use in problem solving and reasoning tasks (Humphreys & Parker, 2015). Teachers can engage in specific discourse practices to encourage students' "math talk;" asking "why?" and "how do you know?" within and outside of designated "number talks" are ways effective teachers use to encourage students to explain their thinking, solve problems, and share mathematical strategies and ideas with their peers (Clements et al., 2023). Structuring students' Math Talk is a key productive teaching routine that centers student thinking and propels instruction (Thanheiser & Melhuish, 2023).

Math Talk has to take place in a safe and supportive classroom environment if learning is to result. "Building a cohesive classroom community is essential for creating a safe, risk-free environment for effective number talks. Students should be comfortable in offering responses for discussion, questioning themselves and their peers, and investigating new strategies. The culture of the classroom should be one of acceptance based on a common quest for learning and understanding" (Parrish, 2010, p. 10). A nurturing, meaning-making Math Talk Community is developed consciously over time (Fuson & Leinwand, 2023; Parrish, 2010) within a culture of mutual respect and a shared enjoyment of investigation and social interaction that the teacher models and sets expectations for from get-go (Parrish & Dominick, 2016). As Fuson and colleagues (2005) explain: "When teachers adopt the role of learners who try to understand their students' methods (rather than just marking the students' procedures and answers as correct or incorrect), they frequently discover thinking that can provide a springboard for further instruction, enabling them to extend thinking more deeply or understand and correct errors. Note that, when beginning to make student thinking visible, teachers must focus on the community-centered aspects of their instruction. Students need to feel comfortable expressing their ideas and revising their thinking when feedback suggests the need to do so" (p. 228). With its evidence-base in equitable practices and positive outcomes for diverse learners as well as its student-centered focus and community-based learning environment (Fuson, et al., 2009; Fuson & Leinwand, 2023), a Math Talk Learning Community supports culturally responsive teaching, as described by researchers such as Aguirre, Mayfield-Ingram, & Martin (2013), Battey (2013), Gay (2018), and Ladson-Billings (1997). "When classrooms are organized into communities that are designed to encourage academic and cultural excellence, students learn to facilitate their own learning as well as that of their fellow students. This kind of classroom requires careful planning and explicit teaching around social interactions so that students learn to assume leadership for learning, feel comfortable exploring differences of opinion, and accept that they may need help from their classmates in order to be successful. Along the way, students learn to see the classroom and their interactions from more than one perspective so that they can identify potential difficulties that come from assumptions of privilege, the distribution of power (who gets to make the rules), and the assessment of performance and competence" (Kozleski, 2010, p. 3).

How *Math Expressions* Delivers

Math Talks is a fundamental aspect of *Math Expressions*: everyone, including the teacher, is both a teacher and a learner. Teachers use the program's extensive, lesson-by-lesson support to foster the learning community as well as supplement and deepen their own mathematical understanding while guiding and growing the same in their students. Students learn to be helpful, contributing members of a teaching-learning math community as they work and talk together. In a **Math Talk Learning Community** environment, students are made to feel safe, trusted, and validated. In such classrooms, competence and confidence develop hand in hand, and all members take the learning path to mathematical proficiency, together. Students are encouraged to share ideas. They also realize mistakes are part of learning; students come to recognize and appreciate their own and others' mistakes. Within this student-centered community, discourse is the shared way of building understandings and promoting one another's thinking and learning.

Student Leaders play many roles in *Math Expressions* classrooms. They lead many of the vital **Quick Practice** routines that start each math class. They often help other students see how to solve problems. They also manage and distribute **MathBoards** or other learning materials. During whole class or small group Math Talk, Student Leaders help others engage in mathematical thinking in a variety of ways. Coaching, modeling, clarifying, explaining, questioning, and offering feedback are just some of the ways these leaders collaborate with their classmates. A vital role of the classroom teacher and staff is to support and mentor students as they develop their skills as Student Leaders throughout the year.

The opening of the *Math Expressions* **Teacher Edition** includes **background and professional learning for teachers about how to build a successful Math Talk community**. In the beginning, teachers concentrate on building listening skills and eliciting comments from students, including using questions helps students expand their explanations. Teachers and more advanced students early on will model how to formulate explanations.

Beginning the Year

Setting Up a Learning Community

If students have used *Math Expressions* before, ask them to describe important aspects of a math class that uses *Math Expressions*. Include the following points if students do not raise them. If students are new to *Math Expressions*, go over these structures to begin to create a Learning Community for the year.

Building Concepts We understand mathematical aspects of situations (we *mathematize*) by seeing and making math drawings to visualize the mathematics and relate these to our explanations.

MathTalk Students explain their thinking in an instructional conversation focused on the math. Math Talk helps everyone understand better. The teacher monitors and extends student Math Talk as needed, helping students learn to talk directly to each other rather than to the teacher.

A key structure of Math Talk is called **Solve and Discuss**. The teacher selects several students to go to the board and solve a problem, using any method they choose. Students love solving at the board, and the teacher can easily see their solving process. The other students work at the same problem at their desks. Then the teacher asks the students at the board to explain their methods.

An explaining student relates their solution to a math drawing, asks if there are any questions, and responds to questions. Listeners are encouraged to ask questions when they do not understand and also to ask lead-in questions that focus on the mathematics that are key to understanding the solution to the problem. Students can also suggest edits to the explanation or solution. So this is actually a Solve, Explain, Question, and Justify structure called Solve and Discuss for short.

During Solve and Discuss, teachers can stand at the side or back of the classroom to help students interact more directly with each other. Teachers say that it is necessary for them to "bite their tongue" to keep from doing all of the talking; student voices and explanations will emerge if you wait. For new topics, teachers may need to model explaining so that students learn to use new vocabulary. But some students can explain even for a new topic.

Usually only two students will explain their solution during Solve and Discuss because listening to many similar explanations is not fruitful. It is better to go on to the next problem to engage students in a new situation to Solve and Discuss. If there are particularly interesting solutions or someone at the board needs a turn at explaining, more than two students should explain their solutions.

Learning Community—Best Practices

MathTalk Aspire to make your classroom a place where all students listen to understand one another. Explain to students that this is different from just being quiet when someone else is talking. This involves thinking about what a person is saying so that you could explain it yourself or help them explain it more clearly. Also, students need to listen so that they can ask a question or help the explainer. Listening can also help them learn more about a concept.

Unit 1 | The Learning Community | LC1

Research indicates that employing the **Solve, Explain, Question, and Justify** participant structure effectively engages students in meaningful Math Talk. To utilize this method, the teacher selects four or five students using any method they choose; next classmates work on the same problem at their desk; then the teacher chooses two or three students to explain their work; finally, the teacher questions the students to prompt them to explain their method. In *Math Expressions*, the teacher orchestrates collaborative instructional conversations focused on the mathematical thinking of classroom members. Together, students and the teacher use seven responsive means of assistance that facilitate learning and teaching by all (several may be used together) and create a nurturing, sense-making, Math Talk Community.

Learning Community and Math Talk Best Practices call-out features throughout the Teacher Editions provide specific, supportive, and embedded help and coaching for reinforcing and deepening these effective and meaningful environments.

Learning Community

Best Practices | Helping Community When students finish their work early, let them help their classmates. Students like to take on this role and enjoy helping each other. Sometimes students who finish early may become bored, and challenging them to explain math content to others may prevent this.

Learning Community

MathTalk **Best Practices** When using **Solve and Discuss**, choose presenters who used different drawings and solution methods and have the class compare and contrast the methods.

Learning Community

MathTalk **Best Practices** You must direct student math talk for it to be productive. Over time as students become more skilled at discussing their thinking and talking directly with each other, you will fade into the background more. But you will always monitor, clarify, extend, and ultimately make the decisions about how to direct the math conversation so that it is productive for your students.

Effective Questioning

"Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships" (NCTM, 2014, p. 35). While many teachers of mathematics allow students to communicate their mathematical thinking, it is critical that such discussions employ effective questioning techniques that genuinely support increased understanding (Childs & Glenn-White, 2018). Teachers' questions are crucial in helping students make connections and learn important mathematics concepts. Questions are a means of both fostering understandings and evaluating understandings (Hattie et al., 2017)—particularly in the early grades (Stiles, 2016).

Questioning techniques shape learning experiences in significant ways, including how students, particularly students of color, see themselves and their capabilities (Goffney, 2018). Questions can open up student discourse, or they can shut it down by "funneling" students' thinking in a particular direction. Questions that encourage discourse express our genuine curiosity about how students are thinking. They require students to articulate and express, and thereby better understand, their own and one another's ideas" (Humphreys & Parker, 2015, p. 26). The kinds of questions math teachers ask and the kind of support that teachers offer are critical on an affective as well as cognitive level, as questions may either facilitate or undermine students' productive efforts and determine whether students view struggle as a positive endeavor or the source of difficulty and frustration (Warshauer, 2015). Developing appropriate questioning techniques is such an important part of mathematics teaching and assessment that one study found "[a] good question may mean the difference between constraining thinking and encouraging new ideas, and between recalling trivial facts and constructing meaning" (Moyer & Milewicz, 2002, p. 293).

NCTM (2014) recommends that teachers present questions drawing from a research-based framework of types that include the following categories:

- gathering information: recall of facts, definitions, and procedures
- problem thinking: explain, elaborate, or clarify thoughts, including the articulation of steps in solution method or task completion
- making learning visible: discuss mathematical structures and make connections among mathematical ideas and relationships
- reflection and justification: reveal deeper understanding of reasoning and actions, including making arguments for validity of their work.

It is also important, NCTM also points out, that in addition to a variety of question times, teachers employ patterns of questioning, including allotting sufficient response time, that focus on and extend students' current ideas to advance their understanding and sense-making about essential mathematical ideas and relationships.

"[T]he key to purposeful questioning is intentionality. Purposeful questioning will not occur through happenstance. These questions will help guide the unpacking of the mathematical task during individual, small-group, and whole- class discussion. It is imperative to incorporate questions that lead discussions to move beyond focusing on students merely obtaining the correct answer, to discussions that focus on making sense of the problem-solving process. Focusing questions on the problem-solving process helps to enrich student's mathematical experiences by allowing the mathematics to move beyond just 'numbers' and 'formulas' to a beautiful concept that is based upon problem-solving" (Childs & Glenn-White, 2018, p. 15-16).

Questioning is an essential component of effective Math Talk (Fuson & Leinwand, 2023; Fuson et al., 2014; Hufferd-Ackles et al., 2004, 2015). While teachers guide the discourse, criteria for the highest level of Math Talk includes the shift from teachers as sole questioners to teachers and students as co-questioners (Hufferd-Ackles et al., 2004, 2015). Fuson and Leinwand (2023) explain:

> "The changes in the teacher role can help teachers transition to higher levels as they support their students to learn to interact with each other instead of just sitting and listening to the teacher. Changes in questioning help everyone change from focusing only on answers to focusing on how answers were obtained and explaining methods. Questions also change from coming only from the teacher to coming from students. As the Math Talk Classroom develops, some of these questions will be questions students have, whereas others will be questions students pose to help other students notice or understand some concept. Explaining thinking develops as the teacher asks more probing questions and helps students by extending their explanations. A simple "Can you convince us?" and "Who can convince us in a different way?" are powerful follow-ups. Students learn that they are expected to explain as fully as they can, and other students listen carefully and also help with extending explanations" (p. 166).

How *Math Expressions* Delivers

Fostering inquiry as a path to learning mathematics begins with a classroom community where students feel safe to take risks and share ideas. The **Inquiry Learning Path** teaching model places students' questions, ideas, and observations at the center of their learning experience. Students understand that finding answers to questions involves generating ideas and realize that everyone's ideas matter. Questioning is a core component within the Math Expressions Math Talk Learning Community. Throughout the *Math Expressions* curriculum, students are guided and encouraged to ask questions and use the answers to devise thoughtful, informed, strategic approaches to solving problems. **Math Talk** gives students the chance to ask questions and use their **MathBoard** to explain and justify their solutions.

Big Idea Questions organize the *Math Expressions* instructional sequence, guiding both teaching and learning and emphasizing the importance of inquiry.

BIG IDEA QUESTIONS

BIG IDEA 1 Meanings of Multiplication and Division: 5s and 2s

BIG IDEA 2 Patterns and Strategies: 9s and 10s

BIG IDEA 3 Strategies for Factors and Products: 3s and 4s

BIG IDEA 4 Multiply with 1 and 0

Effective questioning is central to building and sustaining a **Math Talk Learning Community**. From the outset of each academic year, teachers are supported in developing best practices for **question-based engagement** and **responsiveness** as part of this larger goal of creating collaborative communities of inquiry.

BIG IDEA 1 Question Where and when can we use multplication and divsion?
BIG IDEA 2 Question What are patterns for multiplying 2, 5, 9, and 10? How are they related?
BIG IDEA 3 Question What are patterns for multiplying 3 and 4? How do these relate to patterns for other numbers?
BIG IDEA 4 Question What are patterns for multiplying 0 and 1? How are they different?

With the effective questioning suggestions and facilitation recommendations found in the *Math Expressions* **Teacher Editions**, teachers are able to focus all the students' attention on the mathematics behind the concept, method, or approach. The questions and dialog help guide students to make connections between their ideas and the core mathematics that support those ideas. The effective questioning strategies provided by *Math Expressions* contribute to achieving this desired level of learning.

Students Discuss and Pose Questions

Have students look at the Table of Contents for their Student Activity Book and discuss the Unit Questions near the top of the page just above Big Idea 1. Tell students that for each unit they will investigate, make sense of, understand and explain, predict, and use the concepts in the real world and in the math world and relate them to other math concepts. Each unit has questions to start this process, but students can ask questions throughout the unit. Read the question aloud with students.

Where and when can we use multiplication or division?

Ask students to share what they know about this question and pose any related questions they have. Tell students that the class will continue to discuss and answer this question and other questions students may ask in future lessons.

There is a new question for each Big Idea that is in the Unit Planning Chart, so tell students that you will discuss a new Big Idea question when you start each Big Idea.

Learning Path Teaching of Mathematics

The Class Learning Path model for teaching mathematics for understanding and fluency that is central to *Math Expressions* is rooted in decades of research conducted by program author Karen Fuson and other leaders in the field, both in the U.S. and internationally. The model is discussed in detail in multiple publications, including Murata, 2013; Murata & Fuson, 2006; Fuson & Murata, 2007; Fuson, 2009; Fuson, et al., 2014; Murata, et al., 2020, which are also the sources for the summary presented here on this page. (Works from other researchers that contributed to the formation of the Learning Path model are cited in the introduction of this paper). The Learning Path model integrates two theoretical frameworks, a Piagetian focus on learning and a Vygotskiian focus on teaching, and connects understanding and fluency via evidence-based, student-generated, accessible methodologies and supports, such as math drawings, productive struggle, and Math Talk, that allow all students to meet rigorous standards and proficiency levels. The aim of this balanced framework is to nurture resourceful and self-regulating problem-solvers using five strands of mathematical proficiency—conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition—that have guided 21st century reform efforts in the United States and Canada.

In the Class Learning Path framework, the Malk Talk Learning Community, as articulated in previous sections of this paper, serves as the safe, structured context for all activity, with all members of the community participating in responsive means of facilitating learning for the group individually and at-large through engaging and involving, managing, and coaching, modelling, clarifying, instructing/explaining, questioning, and providing and receiving feedback. For each math topic taught within a classroom, the Learning Path itself consists of three essential phases:

Phase 1: Guided Introducing (exploration of prior knowledge and current understandings students bring to the topic)

- teacher orchestrates collaborative math talk conversations, with teacher and students valuing and discussing all ideas and methods in accordance with Math Talk Learning community principles and practices.
- teacher identifies various solutions methods and typical errors, which are then discussed by the class.

Phase 2: Learning Unfolding (major meaning-making process)

- teacher helps students develop emergent networks of forms-in action
- students engage in explanations of methods and use math drawings and other pedagogical supports to stimulate correct ways of relating to the forms
- advantages and disadvantages of various methods, including the most common method are explored, compared, and discussed and mathematical aspects becomes explicit
- teacher introduces or brings focus to mathematically desirable and accessible method(s)
- erroneous methods are examined, analyzed, and repaired

- methods employed by students become increasingly mathematically desirable and errors decrease; students may also apply newly acquired, confirmed methods in new problems

Phase 3: Kneading Knowledge (fluency)

- teacher helps students gain fluency with desired method(s)
- students may choose a method; part of fluency includes being able to explain and justify the method
- reflection and explanations continue; math drawing and other supports for the topic are surrendered when no longer needed

Depending on the level of complexity of a given concept, these three phases of the Learning Path may occur over several lessons, or they may occur in one lesson. After all three phases are completed for a given topic, and a review stage begins, fluency is maintained by engaging students in practice distributed over time and also to relate established concepts to new concepts that will be learned later. The teacher may aid students in remembering concepts they have already learned by giving them related problems on occasion as well as initiating and orchestrating instructional discussions that assist re-forming individual internal forms to support and stimulate new forms for related topics. Throughout the three phases of the framework, teachers must maintain high expectations of all students, accept and validate the different ideas students bring as well as the varied learning pathways students may take, and trust that every student will in their own course come to use a mathematically desirable method with varied degrees of fluency.

Within the Class Learning Pathways model and, simultaneously, within the Math Talk Community contextual container, evidence-based practices that have driven 21st century mathematics reform—such as representing, problem solving, reasoning, connecting, and communicating—should be integrated into the teaching of the foundational and achievable goals. These practices "can be paired and named to provide a summary of effective math teaching-learning in action: The teacher and the children are doing math sense- making about math structure using math drawings (visual models) to support math explaining" (Fuson, Clements & Sarama, 2015, p. 68).

Math Expressions ©2026 provides teachers with an intentionally sequenced and fully supported Class Learning Pathways instructional design model. Through this model, which was built upon decades of research, students are provided rich, ongoing opportunities for math sense-making, structure, drawing, and explaining experiences that facilitate mathematics learning and success for all.

Math Sense-Making

Sense-making is integral to teaching mathematics for authentic understanding (Battista, 2017; Fuson, et al., 2014; Silver & Mesa, 2011, NCTM, 2009, 2014; Thanheiser & Melhuish, 2023). Because they together comprise the foundation of mathematical competence and proficiency, reasoning and sense-making should be the primary targets of mathematical instruction. Reasoning is the process of manipulating and analyzing objects, symbols, representations, diagrams, or statements to form conclusions based on evidence or assumptions. In contrast, sense-making is the process of utilizing various strategies and visual aids, and/or engaging in discourse and/or drawing upon prior knowledge or experiences to correctly identify, understand, describe, explain, or apply ideas and concepts or justify solutions (Battista, 2017; NCTM, 2009, 2014). "An excellent mathematics program requires effective teaching that engages students in meaningful learning through individual and collaborative experiences that promote their ability to make sense of mathematical ideas and reason mathematically" (NCTM, 2014, p. 7).

Traditionally within mathematics instruction, problem solving has focused on formal representations and rote procedures. Over the past several decades research and reform efforts have emphasized the equally important and complementary process of sense-making, which is both intertwined with logical reasoning and less formal, more intuitive and personal in nature (Battista, 2017; Beckmann, 2002; Tabachneck, Koedinger, & Nathan, 1994). In studying what they deemed the multiple strategy effect, Tabachneck and colleagues (1994) found that combining formal and informal strategies is more effective than single strategy approaches to doing and learning mathematics: "Formal strategies facilitate computation because of their abstract and syntactic nature; however, abstraction can lead to nonsensical interpretations and conceptual errors. Reapplying the formal strategy will not repair such errors; switching to an informal one may. We explain the multiple strategy effect as a complementary relationship between the computational efficiency of formal strategies and the sense-making function of informal strategies" (abstract).

Engaging students in problem-solving tasks allow them to actively construct mathematical understandings more deeply and with greater meaning than when teachers simply present information to students and have them carry out procedural exercises (Masingila, Olanoff, & Kilmani, 2018). "To be effective, mathematics teaching must carefully guide and support students as they attempt to construct personally meaningful mathematical ideas in the context of problem solving, inquiry, and student discussion of multiple problem-solving strategies. This sense-making and discussion approach to teaching can increase equitable student access to powerful mathematical ideas, as long as it regularly uses embedded formative assessment to determine the amount of guidance each student needs" (Battista, 2017, p. 2). Purposeful questions focused on problem solving and that serve to guide discussions (whole class and small group) help to advance students sense-making and reasoning about important mathematical ideas and relationships; in addition to a variety of question times, teachers should employ patterns of questioning, including allotting sufficient response time, that move students beyond merely seeking to obtain a correct answer and through the sense-making

process in ways that extend students' current ideas and advance their understanding (Childs & Glenn-White, 2018; NCTM, 2014).
Math drawings are powerful semiotic tools that aid thinking and the construction of understanding in highly personal ways; such visual renderings support sense-making for individual students or within classroom discourse, especially when teachers and students explain the drawings they make, as explaining is also an integral part of the mathematics sense-making process (Fuson, 2020; Fuson et al., 2014; Murata et al. 2020; Stylianou, 2011). "Certainly, making sense of mathematics and engaging in mathematical reasoning are intimately connected to explaining mathematics. Every mathematics teacher knows that when we explain mathematics, we enhance and solidify own understanding of mathematics. And every mathematics teacher knows that when we explain (or prepare to explain) mathematics, we sometimes uncover our own lack of understanding. It is only when we can explain a piece of mathematics in a way that makes sense both logically and intuitively that we feel we understand the mathematics" (Beckmann, 2002, p. 1)

Students who have opportunity to explore, to solve an unfamiliar problem or devise their own formulas and approaches to an unfamiliar problem in advance of receiving teacher-directed instruction typically demonstrate more positive gains and outcomes (Rittle-Johnson, 2017). "[P]eople who believe they are capable of making sense of unfamiliar things often succeed because they invest more sustained effort in doing so" (NRC, 2012, p. 89). When students are given the time and space to actively construct meaning through applying their own strategies and prior knowledge, and in a process that includes grappling, occasional frustration even, as well as errors, they not ultimately show increased understanding on measures of learning, they also develop traits of productive perseverance essential to school and life (Hiebert & Grouws, 2007; Kapur, 2010, 2014; NRC, 2001, 2012; Warshauer, 2015)
"Developing a productive disposition requires frequent opportunities to make sense of mathematics, to recognize the benefits of perseverance, and to experience the rewards of sense making in mathematics" (NRC, 2001, p. 131).

Mathematical discourse is a critical means through which students make sense of mathematics; the experience of communicating and co-constructing ideas, strategies, and procedures develops deeper understanding as well as more positive dispositions and feelings towards math (Morgan et al., 2014; Leinwand & Fleischman, 2004; Michaels, et al., 2008; Smith & Stein, 2011/18). For English learners, scaffolding tasks and amplifying language to support students in math sense-making promotes language and content development in tandem (Zwiers, Dieckmann, Rutherford-Quach, Daro, Skarin, Weiss, & Malamut, 2017). A nurturing Math Talk classroom environment has been shown to foster sense-making. The steps students invent in the process of sense-making are often initially quite fragile; therefore, in the course of Math Talk, teachers should listen and attend to such individual and internal forms and help students, independently and collectively, bridge understanding and adapt their own sense-making into the external and formal (Fuson, et al., 2014). Within effective math talk communities, students increasingly assume responsibility for the learning and evaluation of others and themselves—and math sense becomes the primary criterion for that evaluation (Fuson & Murata, 2007; Murata et al., 2020).

How *Math Expressions* Delivers

Building Concepts in the classroom requires experiences in which students use objects, drawings, conceptual language, and real-world situations—all of which help students build mathematical ideas that make sense to them. It is necessary, then, for these meaningful supports to be linked to formal mathematics notation, language, and methods. Once this link is solid, students will find meaning in formal mathematics. Drawing on decades of research, Math Expressions effectively aids teachers in helping students build this connection between informal mathematical ideas and formal mathematics, largely through math language and drawings to:

- Develop conceptual understanding
- Support making sense of quantities and their relationships in problem situations
- Allow students to make generalizations and reason abstractly

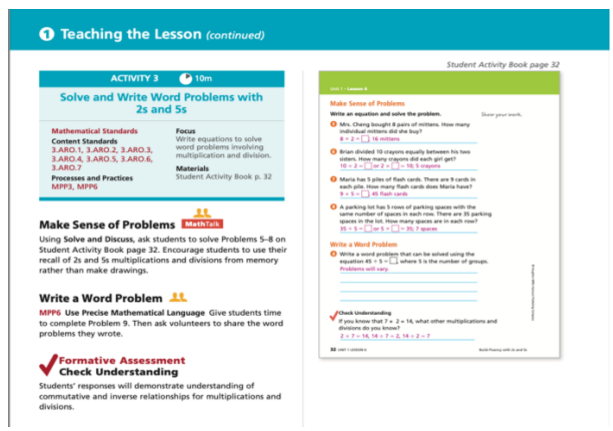
Also central to this approach within Math Expressions is creating and growing a **nurturing sense-making Math Talk Community.** The teacher orchestrates collaborative instructional conversations focused on the mathematical thinking of students, where all strategies are valued and recognized. Teacher actions supported by the program include:

- Engaging and involving
- Managing
- Coaching (via modeling, cognitive restructuring/clarifying, instructing/explaining, questioning, feedback)

The teacher supports the sense-making of all classroom members by using and assisting students to use and relate:

- Coherent mathematical situations
- Pedagogical supports
- Cultural mathematical symbols and labels

Sense-making is an essential component within the Math Talk classroom experiences cultivated by *Math Expression* as well as the student learning activities included throughout the program, which are supported by the Teacher Editions.



Math Structure

There are some mathematical skills which may be best developed with practice in the context of a meaningful examination of patterns and strategies (Fuson, 2009) and recognizing structures (when they do and do not hold) is important mathematical practice (Bofferding & Hansen, 2022; Fuson, 2020; NCTM, 2014). Essentially all math is based on patterns and structure—and young children's mathematical reasoning develops in response to noting patterns and structures in the world around them as well as in symbolic objects they encounter (Mulligan & Mitchelmore, 2013). As defined by Mason, Stephens, and Watson (2009), mathematical structures are the discernable, general properties represented by or across elements with particular concepts, situations, sequences, or relationships and structural thinking is the disposition to observe, use, connect, apply, and explicate such properties within mathematical processes. Structures may be spatial, numerical, or based on data modeling (Mulligan & Mitchelmore, 2013) and drawn from various aspects of mathematics including but not limited to objects, counting, arithmetic, multiplicative patterns, quadratics, sets with functions among them, and geometry (Mason et al., 2009). Structural awareness entails observing and evaluating repetitions and growing patterns, structured counting and grouping, grids and shapes, partitioning, additive and multiplicative structures, measurement and data, and transformations (Mulligan, Oslington, & English, 2020).

Research has long documented deficiencies in U.S. students' conceptual structures within mathematics learning and the challenges that schools face in developing them in students (Bofferding & Hansen, 2022; Fuson, 1990; Fuson, Wearne, Hiebert, Murry, Human, Olivier, Carpenter, Fennema, 1997; Mason et al., 2009; Mulligan & Mitchelmore, 2013; Mulligan et al., 2020)—as well as the barriers to success in higher education and professional life experienced by those who complete K-12 lacking deeper understanding to facilitate flexible reasoning in mathematics, instead continuing to rely on previously memorized and often incorrect procedures (Richland, Stigler, & Holyoak, 2012). Mason and colleagues (2009) contend that cultivating an appreciation of and attentiveness to structure in the youngest students is essential for sustaining their interest in and motivation to learn math as well as enabling their capacity to think productively, deeply, and meaningfully about math. "Research indicates that discovering patterns or relations facilitates mastery with fluency…. Focusing on structure, rather than memorizing individual facts by rote, makes the learning, retention, and transfer for any large body of factual knowledge more likely" (Baroody, Bajwa, & Eiland, 2009, p. 70).

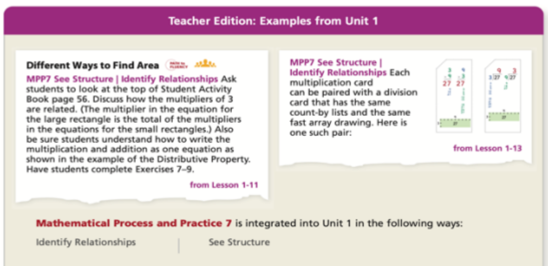
Because effective mathematical reasoning entails the capacity to observe patterns and structures, when students apply structural thinking instead of automatically following procedures and algorithms, they will likely make fewer errors as well as develop more efficient and elegant strategies (Lucenta & Kelemanik, 2022). "[M]astering procedures is an important component of taking advantage of opportunities to make mathematical sense, but that it is of little value to learners if it is simply a procedure, because as the number of procedures increases, the load on memory and retrieval becomes more and more burdensome. When procedures are accompanied by even a minimal appreciation of the mathematical structures which make them effective and which provide criteria for appropriateness, learning shifts to focusing on re-construction based on remembering (literally) rather than relying totally on photographic or rote memory" (Mason et al., 2009, p. 11).

Math Talk has also been found to build students capacity to recognize structure (Clements et al., 2023). Language itself provides a framework for identifying and understanding mathematical structures (Fuson, 1990), and so, as is the case across aspects of math learning, discourse is a highly effective means of developing students' structural thinking, especially when accompanied by visual representations (NCTM, 2014). Physical manipulation of sensory-concrete objects, such as pattern blocks or building blocks, can support awareness of mathematical structures and provide opportunities to investigate and develop structural thinking and processing (Clements et al., 2023; Fuson, 1990; Mason et al. 2009, NCTM, 2014). Discursive approaches and questioning should include discussing similarities among representations revealing underlying mathematical structures to make connections among mathematical ideas and relationships and having students explain essential features of mathematical ideas that persist regardless of form (Zimba, 2011, NCTM, 2014) as well as analyzing worked examples and trying to make sense of them by drawing on prior knowledge has also been found to build students capacity to recognize structure (Clements et al. 2023). Fuson (with Murata, 2007; Murata, & Abrahamson, 2014; Clements, & Sarama, 2015) urges teachers to have students" do math sense-making about math structure using math drawings to support math talk" (p. 11).

Similar to classroom management routines, mathematical thinking routines offer a predictable set of actions that students learn and then practice repeatedly until internalized (Kelemanik, Lucenta, & Creighton, 2016; Thanheiser & Melhuish, 2023). A coherent math curriculum is sequenced within and across grade levels in a way that best reflects the hierarchical and logical structures of mathematics (Schmidt, Wang, & McKnight, 2005). Mathematics programs that effectively support the development of essential 21st century skills are organized by coherent learning progressions reflecting broad levels of learning; they aid students in recognizing and analyzing patterns of thinking, including concepts and structures, and constructing conceptual understandings as well as connections among areas of mathematical study and between mathematics and the real world (Clements et al., 2023; NCTM, 2014). Mathematical proficiency and deep levels of learning requires carefully structured, interactive instructional activities that feature predictable and repeatable routines that allow students to focus on and engage with tasks, content, problems, and each other (Lampert, 2015; Thanheiser & Melhuish, 2023). "Mathematical proficiency depends also on other mental habits that dispose one to characterize problems (and solutions) in precise ways, to subdivide and explore problems by posing new and related problems, and to 'play' (either concretely or with thought experiments) to gain experience and insights from which some regularity or structure might be derived" (Goldenberg, Mark, Kang, Fries, Carter, & Cordner, 2015, p. 1-2).

How *Math Expressions* Delivers

Throughout *Math Expressions* teachers continually assist students **to do math sense-making about math structure using math drawings to support math explaining**. The program's **Teacher Editions** embed within Math Talk and lesson content ongoing opportunities as well as guidance for when and how to help students see mathematical structure and identify relationships and patterns.



Teacher Edition: Examples from Unit 1

Different Ways to Find Area

MPP7 See Structure | Identify Relationships Ask students to look at the top of Student Activity Book page 56. Discuss how the multipliers of 3 are related. (The multiplier in the equation for the large rectangle is the total of the multipliers in the equations for the small rectangles.) Also be sure students understand how to write the multiplication and addition as one equation as shown in the example of the Distributive Property. Have students complete Exercises 7–9.

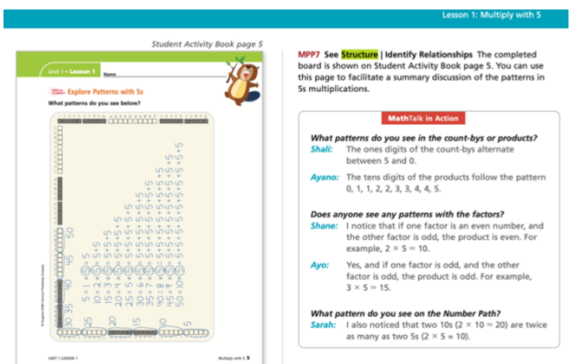
from Lesson 1-11

MPP7 See Structure | Identify Relationships Each multiplication card can be paired with a division card that has the same count-by lists and the same fast array drawing. Here is one such pair:

from Lesson 1-13

Mathematical Process and Practice 7 is integrated into Unit 1 in the following ways:

Identify Relationships | See Structure



Lesson 1: Multiply with 5

Student Activity Book page 5

Explore Patterns with 5s

What patterns do you see below?

MPP7 See Structure | Identify Relationships The completed board is shown on Student Activity Book page 5. You can use this page to facilitate a summary discussion of the patterns in 5s multiplications.

MathTalk in Action

What patterns do you see in the count-bys or products?

Shali: The ones digits of the count-bys alternate between 5 and 0.

Ayano: The tens digits of the products follow the pattern 0, 1, 1, 2, 2, 3, 3, 4, 4, 5.

Does anyone see any patterns with the factors?

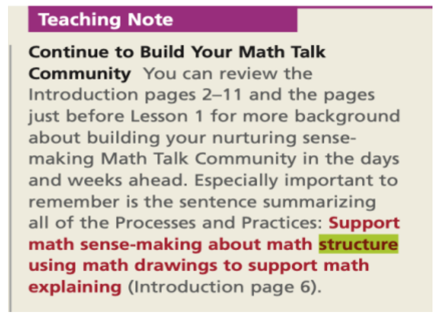
Shane: I notice that if one factor is an even number, and the other factor is odd, the product is even. For example, $2 \times 5 = 10$.

Ayo: Yes, and if one factor is odd, and the other factor is odd, the product is odd. For example, $3 \times 5 = 15$.

What pattern do you see on the Number Path?

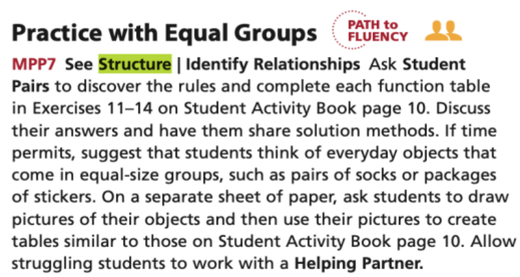
Sarah: I also noticed that two 10s ($2 \times 10 = 20$) are twice as many as two 5s ($2 \times 5 = 10$).

Recognizing structure is integrated with math learning and among the key **mathematical processes and practices** that *Math Expressions* cultivates, habituates, and sustains within the **Math Talk Learning Community** environment.



Teaching Note

Continue to Build Your Math Talk Community You can review the Introduction pages 2–11 and the pages just before Lesson 1 for more background about building your nurturing sense-making Math Talk Community in the days and weeks ahead. Especially important to remember is the sentence summarizing all of the Processes and Practices: **Support math sense-making about math structure using math drawings to support math explaining** (Introduction page 6).



Practice with Equal Groups PATH to FLUENCY

MPP7 See Structure | Identify Relationships Ask **Student Pairs** to discover the rules and complete each function table in Exercises 11–14 on Student Activity Book page 10. Discuss their answers and have them share solution methods. If time permits, suggest that students think of everyday objects that come in equal-size groups, such as pairs of socks or packages of stickers. On a separate sheet of paper, ask students to draw pictures of their objects and then use their pictures to create tables similar to those on Student Activity Book page 10. Allow struggling students to work with a **Helping Partner.**

Math Drawing

Because mathematics entails the use of signs, symbols, and diagrams to represent abstract notions and study spatial aspects, as well as because the nature of the subject is often invisible and intangible, visualization is integral to learning and teaching mathematics (Presmeg, 2020; Bobis & Way, 2018; Fuson, et al., 2015; Stylianou, 2011). A wide body of research supports the use of physical and imagistic models, drawings, manipulatives, and other such representations in the mathematics classroom; such representations help make abstract concepts more concrete as well as aid in internalization of procedures for problem-solving, increased creativity, greater metacognition, and students' more active participation in their own learning—all of which contribute key elements for impactful mathematical exploration. Indeed, the abstract nature of mathematics means that people have access to mathematical ideas largely or often only through the representations of those ideas (Carbonneau & John-Steiner, 2013; Cross et al., 2009; Fuson et al., 2014; NCTM, 2000 & 2014; NRC, 2001; Stylianou, 2011). Representations, models, and manipulatives are important in math learning for all students, but significant research indicates that younger students and those who have special needs or are English learners or those having difficulty grasping abstract mathematical concepts especially benefit from visual representations of mathematical ideas, including physical objects they use or actions they perform when trying to solve problems (Agrawal & Morin, 2016; Bouck & Park, 2018; Flores, 2010; Fuson, 2019; Fuson et al., 1997; Fuson et al., 2009; Miller & Hudson, 2007; NRC, 2001; Riccomini, Witzel, & Deshpande, 2022; Sousa, 2008), especially when accompanied by explanations and discussions of the images among students and teachers (Fuson et al., 2009; Fuson, et al., 2014; Murata et al., 2020)

Mathematical representation is commonly thought to be a product—a picture or a set of symbols students make to demonstrate understanding; however, representation in math learning is also an essential cognitive process. Students' diagrams and symbolism evolve dynamically over the course of problem-solving and aid thinking and the construction of understanding in highly personal ways (Stylianou, 2011). Representations "help to portray, clarify, or extend a mathematical idea by focusing on its essential features" (NCTM, 2000, p. 206). Research has shown that visualization is a powerful problem-solving tool and can be helpful in all kinds of mathematical problems, not only geometric problems (Van Garderen, 2006) and that using visual representations improves student performance at all levels of math education, from general mathematics, prealgebra, word problems, and operations (Gersten, Beckmann, Clarke, Foegen, Marsh, Star, & Witzel, 2009). "For students to understand such mathematical formalisms, we must help them connect these formalisms with other forms of knowledge, including everyday experience, concrete examples, and visual representations. Such connections form a conceptual framework that holds mathematical knowledge together and facilitates its retrieval and application" (Kalchman & Koedinger, 2005, p. 364). When students sketch or visually organize their mathematical thinking, they are able to explore their understanding of concepts, procedures, and processes—and communicate mathematically (Arcavi, 2003; Stylianou & Silver, 2004). Representations bolster intuition and understanding (Blatto-Vallee, Kelly, Gaustad, Porter, & Fonzi, 2007) and can help students to communicate, reason, problem solve, connect, and learn (Hill, Sharma, Obyrne, & Airey, 2014). Having students additionally explain and participate in discussions about their representations allows for more meaningful learning (Fuson, 2020; Fuson, 2019; Fuson et al., 2009; Fuson, Smith, & Cicero, 1997; Fuson & Murata, 2007; Murata et al., 2020). "[T]eachers should help children relate work with quantities and objects in the real world to pictures of these in books and on other two-dimensional surfaces or pieces of paper...In later preK and in kindergarten, it is also important to help children relate real three-dimensional objects to math drawings of these objects and to written mathematical symbols. At these ages, all children need both kinds of experiences: with actual objects and things and with math drawings and symbols of the things...The vital issue is to support meaning making, mathematizing, and making connections among mathematical language, symbols, and quantities and shapes" (Fuson, et al., 2015, p.67).

A trajectory that progresses students through instruction aiming to develop first concrete, then representational or pictorial, and then, ultimately, abstract understanding—also known as CRA or CPA—has long been widely practiced within the field of mathematics education. The theoretical basis of the framework derives from Bruner (1966), who proposed that the learning sequence of a new mathematical concept begins with concrete actions undertaken or with concrete objects (enactive), which is then translated into perceptual images of the experience formed (iconic), subsequently leading to the adoption of the mathematical symbol. The three phases of CRA typically entail use of concrete materials (such as blocks, tiles, cubes, base ten blocks, fraction strips, etc.), then two-dimensional written/pictorial representations and notations (including circles, dots, etc.), and finally abstractions in the form of symbols, numbers, equations, and algorithms; this research-based approach has been shown to generate meaningful connections, foster fluency, and support conceptual understanding of mathematics (Flores, 2010). This sequential progression is particularly effective for students who are at-risk or have mathematics difficulties, mainly because it moves gradually from actual objects through pictures and then to symbols (Agrawal & Morin, 2016; Flores, 2010; Sousa, 2008). A key element within the CPA approach is the systematic use of manipulatives—concrete, sensory materials such as blocks, cubes, base-ten blocks, place value cards, fraction strips, etc. or such objects in digital form—as scaffolds within explicit instruction focused on developing conceptual understanding and mathematical knowledge (Bouck & Park, 2018; Clements & Sarama, 2007; Fuson, 2019; Miller & Hudson, 2007). "When used in comprehensive, well-planned, instructional settings, both physical and virtual manipulatives can encourage students to make their knowledge explicit, which helps them build Integrated-Concrete knowledge" (Sarama & Clements, 2016, p. 71). However, Fuson, Clements and colleagues (e.g. 2015, 2023) stress that, while perceptual supports are necessary and fundamental in helping students investigate and understand mathematical structures and processes while they progress from sensory-concrete to integrated concrete, cognition stages, once students learn explicit generalizations and abstractions that they can verbalize, indicating the developments of more sophisticated mental imagery, it is critical to transition students away from concrete manipulative and into the use drawings and symbols, so this initial stage of support may be short or even non-existent, depending on the needs of an individual student.

A nurturing Math Talk classroom marked by an inclusive and participatory culture and language-the rich environment supports and is supported by sense-making, particularly math drawings generated from both students and teachers (Fuson, 2020; Fuson & Murata, 2007; Fuson et al., 2014; Murata et al., 2020. Math drawings are powerful semiotic tools as well as a form of visual aid that support sense-making both for an individual student and in collaborative classroom discourse about mathematical thinking (Fuson, 2020; Fuson, et al., 2009; Murata et al., 2020).

How *Math Expressions* Delivers

Throughout their engagement with *Math Expressions*, students develop an understanding of the concepts behind the math and build connections among concrete, pictorial, and abstract relationships. **Math drawings** are a core, ongoing, integrated learning practice within the *Math Expressions'* **Math Talk** classroom. Students not only continually generate their own math drawings, they also recognize the utility of such drawings as an essential tool for understanding and communicating mathematical concepts and ideas. Indeed, within *Math Expressions,* both students and teachers use **math drawings** as tools for teaching and learning. The value of math drawings to make sense and explain is promoted explicitly and experientially.


Unit 1 • Lesson 2

Make a Math Drawing to Solve Problems

Make a drawing for each problem. Label your drawing with a multiplication equation. Then write the answer to the problem.

Show your work.

Possible drawings

4 Sandra is carrying 4 bags of lemons. There were 6 lemons in each bag. How many lemons did she buy altogether?

24 lemons

$4 \times 6 = 24$

5 Batai baked 2 peach pies. He used 7 peaches per pie. How many peaches did he use altogether?

14 peaches

$2 \times 7 = 14$

6 The Fuzzy Friends pet store has 3 rabbit cages. There are 5 rabbits in each cage. How many rabbits does the store have altogether?

15 rabbits

$3 \times 5 = 15$

7 The Paws Plus pet store has 5 rabbit cages. There are 3 rabbits in every cage. How many rabbits does the store have altogether?

15 rabbits

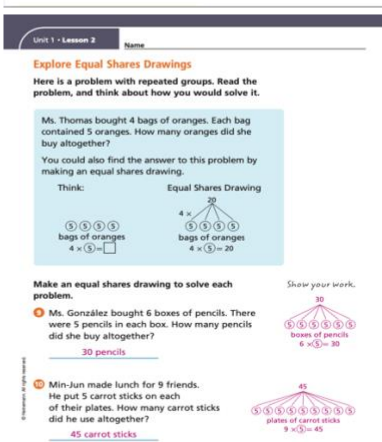
$5 \times 3 = 15$

8 There are 7 players on an ultimate frisbee team. The team wants to order 2 game shirts for each player. How many game shirts does the team need to order altogether?

14 shirts

$7 \times 2 = 14$

8 UNIT 1 LESSON 2 Multiplication as Equal Groups


Unit 1 • Lesson 2 Name

Explore Equal Shares Drawings

Here is a problem with repeated groups. Read the problem, and think about how you would solve it.

Ms. Thomas bought 4 bags of oranges. Each bag contained 5 oranges. How many oranges did she buy altogether?

You could also find the answer to this problem by making an equal shares drawing.

Think: bags of oranges $4 \times 5 = \square$

Equal Shares Drawing: 20, 4 ×, bags of oranges $4 \times 5 = 20$

Make an equal shares drawing to solve each problem.

Show your work.

9 Ms. González bought 6 boxes of pencils. There were 5 pencils in each box. How many pencils did she buy altogether?

30 pencils

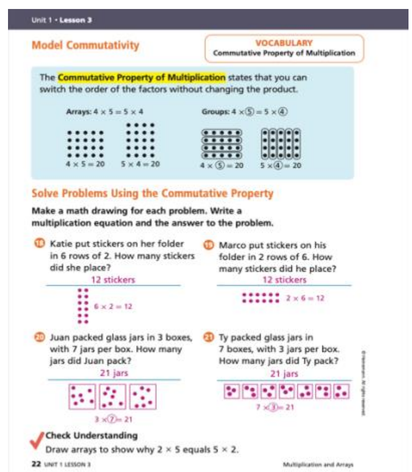
30, boxes of pencils $6 \times 5 = 30$

10 Min-Jun made lunch for 9 friends. He put 5 carrot sticks on each of their plates. How many carrot sticks did he use altogether?

45 carrot sticks

45, plates of carrot sticks $9 \times 5 = 45$

In *Math Expressions,* math drawings focus on the mathematical aspects of quantities or of a problem situation. These sense-making and richly meaningful visual representations are central to lessons in ***Math Expressions*** and are used together with **Math Talk** as students explain their thinking and listen to the explanations of other students. The use of math drawings enables students to work at their own entering level but also moving forward to build intertwined understanding and fluency. Students also engage in math drawing as regular part of their **Math Journals**. Math drawings further provide invaluable ongoing **formative assessment** via **Check Understanding**.


Unit 1 • Lesson 3

Model Commutativity

VOCABULARY
Commutative Property of Multiplication

The Commutative Property of Multiplication states that you can switch the order of the factors without changing the product.

Arrays: $4 \times 5 = 5 \times 4$ — $4 \times 5 = 20$, $5 \times 4 = 20$

Groups: $4 \times 5 = 5 \times 4$ — $4 \times 5 = 20$, $5 \times 4 = 20$

Solve Problems Using the Commutative Property

Make a math drawing for each problem. Write a multiplication equation and the answer to the problem.

18 Katie put stickers on her folder in 6 rows of 2. How many stickers did she place?

12 stickers

$6 \times 2 = 12$

19 Marco put stickers on his folder in 2 rows of 6. How many stickers did he place?

12 stickers

$2 \times 6 = 12$

20 Juan packed glass jars in 3 boxes, with 7 jars per box. How many jars did Juan pack?

21 jars

$3 \times 7 = 21$

21 Ty packed glass jars in 7 boxes, with 3 jars per box. How many jars did Ty pack?

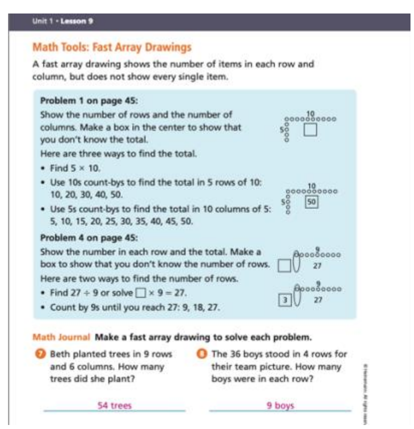
21 jars

$7 \times 3 = 21$

Check Understanding

Draw arrays to show why 2×5 equals 5×2.

22 UNIT 1 LESSON 3 Multiplication and Arrays


Unit 1 • Lesson 9

Math Tools: Fast Array Drawings

A fast array drawing shows the number of items in each row and column, but does not show every single item.

Problem 1 on page 45:

Show the number of rows and the number of columns. Make a box in the center to show that you don't know the total.

Here are three ways to find the total.

- Find 5×10.
- Use 10s count-bys to find the total in 5 rows of 10: 10, 20, 30, 40, 50.
- Use 5s count-bys to find the total in 10 columns of 5: 5, 10, 15, 20, 25, 30, 35, 40, 45, 50.

10, 5, 50

Problem 4 on page 45:

Show the number in each row and the total. Make a box to show that you don't know the number of rows.

Here are two ways to find the number of rows.

- Find $27 \div 9$ or solve $\square \times 9 = 27$.
- Count by 9s until you reach 27: 9, 18, 27.

9, 27, 3

Math Journal **Make a fast array drawing to solve each problem.**

7 Beth planted trees in 9 rows and 6 columns. How many trees did she plant?

54 trees

8 The 36 boys stood in 4 rows for their team picture. How many boys were in each row?

9 boys

Across the *Math Expressions* curriculum, all grade levels include activities involving use of the program's **MathBoards**. Students use a **MathBoard** for representing and solving problems, recording and sharing their thinking, relating their drawings to math language, and justifying their thinking. At the primary grades and for other students with emergent concept development and skill levels, *Math Expressions* also utilizes manipulatives. and other tools that serve as visual representation aids. These include **Secret Code Cards**, **Count-On Cards**, **2- and 3-Dimensional Shape Sets**, and **Base Ten Blocks**.

Math Explaining

Within the balanced learning pathways approach to teaching developed by Fuson, Murata, and Abrahamson (2014 & 2020), mathematically desirable methods that are accessible to all students—possibly even invented by students—are connected and explained using visual referents, especially math drawings, to support meaning making; this approach enables all students to use general methods with understanding and process to fluency. Additionally, a major focus of the National Science Foundation-funded research that is the foundation of *Math Expressions* was identifying ways to help students understand and be able to explain word problems and computational methods, while also bringing students to accuracy and fluency in these areas (Fuson, 2018).

Generating explanations to make sense of new information as well as explanations of solutions to math problems in ways that connect new information to background knowledge are cognitive activities that promote the development of conceptual knowledge, procedural knowledge, and procedural flexibility (Rittle-Johnson, 2017). Lampert (2015) recommends that having students explain their thinking to one another become a standard component of a predictable classroom routine, as it cultivates desirable mathematical habits. "Students who learn to articulate and justify their own mathematical ideas, reason through their own and others' mathematical explanations, and provide a rationale for their answers develop a deep understanding that is critical for future success in mathematics" (Carpenter, Franke, Levi, 2003, p. 4). Math talk is an essential component of mathematical thinking and to be effective math talkers, students need to develop skills across the components of questioning, explaining mathematical thinking, identifying the source of mathematical ideas, taking responsibility for learning, and mathematical representations (Cuoco et al., 1996; Thanheiser & Melhuish, 2023). "Even the clearest teacher explanations leave many students with incomplete understanding and shaky confidence. Ideas that are forged by hard thought and tested in discourse with other students and teachers are much more likely to last and be useful" (Marcus & Fey, 2003, p. 61).

Indeed, explanation is an integral component within nurturing Math Talk Communities. As outlined by Hufferd-Ackles and colleagues (2015), one of the ways to evaluate the effectiveness of a given community is to determine who provides explanations in the classroom and what kinds of explanations are offered. The aim is to move the explaining of mathematical thinking away from an environment in which teachers do all the questioning with a focus on correctness to teachers to probing strategically and increasingly deeply to learn more about students' thinking via students' descriptions of their thinking. Ultimately, in a fully and authentically realized Math Talk Learning Community, the teacher follows student explanations and asks students to compare and contrast strategies, and the students volunteering explanations of their thinking and justifying their answers.

As demonstrated in the Children's Math World research that is the foundation of *Math Expressions*, within effective Math Talk communities, however, the explanations are inextricably linked with "conceptual supports to help make mathematics personally meaningful to students and through which students can share their ideas with others. In this curriculum, students make mathematical drawings to solve problems and explain their thinking, and they label these math drawings and related equations to link them to the problem situation" (Hufferd-Ackles et al., 2015, p. 126). Math drawings facilitate problem solving by bridging problem situations with mathematical solutions through mathematizing. Math drawings also assist instructional conversations as they can include all notations of steps in the thinking and then be displayed for all to see, examine, and explain. In these ways, the drawings support reflection and further explanation (Fuson et al., 2014; Murata et al., 2020). A Math Talk Learning Community fosters mathematical knowledge and skill building, including mathematical explanations, while reinforcing conceptual understanding. This is true for all students and affords particular benefits for students with disabilities. Fuson (2019) describes these mutually supportive processes: "A strength of the conceptual web connected by Math Talk is that most children have at least one modality in which they are strong enough and that can help support the other modalities that need to become related. So explaining may be difficult for one child, but she perhaps can make drawings of her thinking or use concrete objects to show her thinking. Then with enough modelling of explaining by others, her own explaining will also improve. Another child might have difficulties with drawing, but can explain or enact with concrete materials. And gradually drawing can improve. These conceptual webs use the power of geometric and spatial images and concepts to support learning (p. 120)."

Other strategies, particularly questioning, promote the development of math explaining. NCTM (2014) recommends that teachers present questions drawing from a research-based framework of problem thinking—specifically, explaining, elaborating, or clarifying thoughts, along with articulation of steps within a solution method or task completion. Effective teachers routinely ask "why?" and "how do you know?" to encourage students to explain their thinking, solve problems, and share mathematical strategies and ideas with their peers (Beckmann, 2002; Clements & Sarama, 2004 & 2007). Noting the recursive, mutually supportive processes of sense-making and explaining, Beckmann (2002) recommends that teachers connect reasoning with ordinary sense-making in ways that are additionally, authentically explanatory by having students generate explanations with the following features: the explanation should be logical and convincing for its audience (of a particular grade or proficiency level) and include multiple, coordinated forms of explanation (such as numerically and visually).

How *Math Expressions* Delivers

Within the *Math Expressions* **Inquiry Learning Path**, students participate in **math sense-making** about **math structure** using **math drawings**—all in support of math explaining. **Explaining math thinking**, along with questioning, contributing math ideas, and taking responsibility for learning, is one of four essential components of **Math Talk** in a *Math Expressions* classroom.

UNIT 6 | Write Equations To Solve Word Problems

BIG IDEA 1 - Types of Word Problems

Mathematical Standards 3.ARO.3, 3.ARO.4, 3.PVO.1, 3.PVO.2, 3.PVO.5

Within this structured environment, students have continual opportunity to engage in mathematical explanations and through this sustained practice over time they learn to effectively communicate their mathematical understanding verbally and in writing. **Math explanation** within *Math Expressions* encompasses reasoning, explaining, and questioning. The program guides teachers in employing the **Solve, Explain, Question, and Justify** participant structure as a means of facilitating meaningful Math Talk:

- The teacher selects four or five students using any method they choose.
- Classmates work on the same problem at their desk.
- The teacher chooses two or three students to explain their work.
- The teacher questions the students to prompt them to explain their method.

Unit 3 • Lesson 5 Name

VOCABULARY
estimate
round

Estimate

Solve the problem.

1. Tasha read 3 books over the summer. Here is the number of pages in each book:

Watership Down	494 pages
Sounder	128 pages
The Secret Garden	368 pages

About how many pages did Tasha read? Explain how you made your estimate.

Answers will vary. Possible answer: About 1,000 pages; I rounded 494 to 500, 128 to 100, and 368 to 400. Then I added the rounded numbers.

Practice Rounding

Round each number to the nearest hundred. Use drawings or Secret Code Cards, if they help you.

2. 128 100	3. 271 300	4. 376 400
5. 649 600	6. 415 400	7. 550 600
8. 62 100	9. 1,481 1,500	10. 2,615 2,600

11. Explain Your Thinking When you round a number to the nearest hundred, how do you know whether to round up or round down?

Answers will vary. Possible answer: If the number is more than halfway to the next hundred, I round up. If it is less than halfway to the next hundred, I round down. If it is exactly halfway between two hundred numbers, I round up.

UNIT 3 LESSON 5 Round to the Nearest Hundred 195

Math explaining is a key facet of student and teacher activity within every lesson within *Math Expressions.*

Teacher Edition: Examples from Unit 1

MPP3, MPP6 Use and Evaluate Logical Reasoning/ Use Precise Mathematical Language | Puzzled Penguin Ask Student Pairs to read the Puzzled Penguin problem on Student Activity Book page 25. Give students an opportunity to complete Problem 5 and write an answer to Puzzled Penguin. Remind students they need to describe Puzzled Penguin's error and explain why it is wrong. In this exercise, Puzzled Penguin used an incorrect factor in the related multiplication.

from Lesson 1-4

MPP3, MPP6 Use and Evaluate Logical Reasoning/ Use Precise Mathematical Language | Puzzled Penguin After students read the letter from the Puzzled Penguin, give them time to write a response. Then allow several students to share ideas about what the Puzzled Penguin did wrong. Encourage them to make drawings on the board if it helps them explain.

See the **Math Talk in Action** on the next page for a sample classroom dialogue.

from Lesson 1-12

Mathematical Process and Practice 3 is integrated into Unit 1 in the following ways:

Puzzled Penguin | Evaluate Logical Reasoning | Justify Conclusions

Math explaining is also an integral part of ongoing **formative assessment**, including via the **Check Understanding** feature, as well as **summative** and **performance assessment** within *Math Expressions*.

Check Understanding

Explain how patterns in the 2s count-bys and multiplications can help you when multiplying.

Mathematical Habits of Mind

In their seminal article, Cuoco and colleagues (1996) proposed that “more important than specific mathematical results are the habits of mind used by the people who create those results…. This includes learning to recognize when problems or statements that purport to be mathematical are, in truth, still quite ill-posed or fuzzy; becoming comfortable with and skilled at bringing mathematical meaning to problems and statements through definition, systematization, abstraction, or logical connection making; and seeking and developing new ways of describing situations” (p. 376). Mathematical habits of mind reflect how mathematicians think about and through complex problems. Proficiency in mathematics requires students to engage in productive mindsets, adopt strategic approaches, and persist toward finding solutions. As Goldenberg et al. (2015) explain:

> The ability to solve new and unforeseen problems requires mastery not just of the results of mathematical thinking (the familiar facts and procedures) but of the ways that mathematically proficient individuals do that thinking. This is especially true as our economy increasingly depends on fields that require mathematics. Mathematical proficiency depends also on other mental habits that dispose one to characterize problems (and solutions) in precise ways, to subdivide and explore problems by posing new and related problems, and to “play” (either concretely or with thought experiments) to gain experience and insights from which some regularity or structure might be derived" (p. 1–2).

Beyond scholastic achievement in mathematics, mathematical habits of mind are essential to critical thought, college and career readiness, access to future opportunities, and productive participation in society (Goldenberg et al., 2015). “If we really want to empower our students for life after school, we need to prepare them to be able to use, understand, control, modify, and make decisions about a class of technology that does not yet exist. That means we have to help them develop genuinely mathematical ways of thinking” (Cross et al., 2009, p. 21).

To cultivate mathematical habits of mind, teachers must also then create a classroom culture that demonstrates how struggle is a natural part of the learning process (Star, 2015) and allows students to see the benefits of perseverance (Hiebert & Grouws, 2007). Educators and students both must also adopt growth mindsets and positive views on productive struggle. These attitudinal states yield numerous positive affective outcomes and boost academic achievement. A growth mindset, as conceived by Carol Dweck (2006, 2008) within mathematics emphasizes teaching and learning as processes that cultivate mathematical abilities; stresses that success and learning are reflections of effort and not intelligence; and promotes a belief that all students are capable of participating and achieving in mathematics. Society has traditionally valued the math learner who can memorize well and calculate fast, rather than others who possess equal potential but may be deeper, slower, and possibly more creative thinkers. These earlier mindsets have contributed to persistent negative perceptions within mathematics education specifically—and the evolution toward pervasive growth mindsets is necessary if all students are to be successful math learners. Teachers should promote and display a growth mindset by valuing all students’ thinking and efforts while also relying on pedagogical practices such as differentiated tasks, mixed-ability groupings, and praise for students' contributions and perseverance within their mathematical learning. Setting and supporting rigorous expectations and a genuine belief that student effort and effective instruction outweigh "smarts" and life circumstances increase students' opportunities to learn and create a more equitable learning experience. Schools and classrooms that reinforce growth mindset messaging make learning enjoyable and place the focus on that learning rather than on students' performance (Blackwell, Trzesniewski, & Dweck, 2007; Dweck, 2006, 2008; Hiebert, Carpenter, Fennema, Fuson, Human, Murray, Olivier, & Wearne, 1996; NCTM, 2014; Yeager, Walton, & Cohen, 2013). Additionally, promoting self-determination is an important component in classroom instruction aimed at helping all students attain post-academic success and quality of life. Particularly for those with special needs, helping students develop skills associated with self-determination (e.g., planning, self-management, self-awareness, problem-solving, and goal setting) is critical in preparation for experiences within and beyond school (Raley, Shogren, & McDonald., 2018). "[M]athematics teaching is most powerful when it starts with appropriately challenging intentions and success criteria" (Hattie et al., 2017, p. 4).

Math Expressions ©2026 utilizes evidence-based methods to cultivate mathematical habits of mind within all students. The program promotes interest and engagement with problem solving and mathematical thinking and provides ongoing opportunities for students to productively persevere through cognitively demanding tasks to resolution—all within a nurturing Math Talk Learning Community. In philosophy and practice, *Math Expressions* presents challenge and builds confidence to help students become lifelong math thinkers and real-world problem solvers.

Mathematical Thinking

"[S]tudent mathematical thinking is the means to move instruction forward" (Thanheiser & Melhuish, 2023, p. 1). Mathematical conceptual understanding, thinking, and reasoning along with the skills to engage in procedural reliability, fluency, and automaticity are vital capacities for 21st century learners (Granovskiy, 2018). "An excellent mathematics program requires effective teaching that engages students in meaningful learning through individual and collaborative experiences that promote their ability to make sense of mathematical ideas and reason mathematically" (NCTM, 2014, p. 7). Effective teaching and its development of students' mathematical knowledge are the driving forces behind powerful mathematics instruction and deep understanding. Research continually demonstrates that mathematics learning should be focused on engaging students in instructional tasks and interactive practices that promote reasoning, problem solving, and discourse—all with the aim of fostering understanding of mathematical concepts and procedures (NCTM, 2009 & 2014; NRC, 2012).

Authentically student-centered mathematics classrooms are characterized by a culture in which instruction is driven by student mathematical sense-making, conjecturing, and justifying as well as one in which student mathematical thinking is shared publicly and students engage with each other's mathematical thinking (Thanheiser & Melhuish, 2023). Effectively problematizing mathematics emphasizes the process of sense-making—having students think for themselves and explain their thinking while also supported by their teacher, classmates, and math program; to struggle productively; and ultimately to apply their gained knowledge and strategies to new and more complex problems they encounter in the future (Hiebert et al., 1996; Li & Schoenfeld, 2019). Mathematical proficiency, then, requires deep learning. Deep learning requires deep thinking. Deep thinking requires carefully structured, interactive instructional activities that feature predictable and repeatable routines that allow students to focus on and engage with tasks, content, problems, and each other: "[r]epeatedly using practices that support these kinds of activities turns important elements of academic engagement into habits. Through repetition, both teacher and students acquire new intellectual and social skills and dispositions. More importantly, perhaps, both teacher and students acquire new ways of thinking about what it means to teach and learn, and what they are able to accomplish" (Lampert, 2015, p. 17). Effective routines for reasoning include collaborative, discursive work through a process of making sense; analyzing situations; interpreting analyzing, and adapting models; and reflecting on thinking (Lucenta & Kelemanik, 2020; Thanheiser & Melhuish, 2023).

NCTM (2014) urges educators to elicit and use evidence of student thinking to evaluate progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning; it is crucial for teachers of mathematics to know what students know and understand (and do not) and this requires close examination of student work (Fuson et al., 2014; Hattie et al., 2017; Murata et al., 2020). "To focus on student-centered mathematics instruction with an explicit mathematical goal, teachers need to identify the types of student thinking that can be leveraged in whole class discussion ... Instructors need to identify this mathematical thinking in the classroom, make it public if needed, and engage students in discussion around each other's ideas" (Thanheiser & Melhuish, 2023, p. 2). Research continually affirms the crucial role that metacognition, which includes awareness one's strengths and weaknesses and recognition of strategies useful to progress in specific tasks, within both mathematical thinking and achievement in mathematical study (Muncer, Higham, Gosling, Cortez, Wood-Downie, & Hadwin, 2022). "Teaching practices congruent with a metacognitive approach to learning include those that focus on sense-making, self-assessment, and reflection on what worked and what needs improving. These practices have been shown to increase the degree to which students transfer their learning to new settings and events" (NRC, 2000, p. 12). Within how children learn mathematics specifically, "[m]etacognition and adaptive reasoning both describe the phenomenon of ongoing sense making, reflection, and explanation to oneself and others" (Fuson et al., 2005, p. 218). Students' metacognition helps them to apply mathematical learning to new contexts: "By describing their own processes, students can use their reflections to develop flexible prototypes of experiences that can be drawn on in future problem solving" (Lesh & Zawojewski, 2007, p. 770)

Important for teachers to develop and sustain mathematical thinking within their students is for teachers themselves to be mathematical in front of—and with—students (Mason, 2020). Such an approach to math instruction is effectively supported within a Math Talk learning community, in which students and teachers learn collaboratively, co-constructing explanations and solutions (e.g., Fuson et al., 2014; Hufferd-Ackles et al., 2004 & 2015). Students' literacy and language skills can be effectively developed in tandem with their mathematical thinking habits and skills. Zwiers and colleagues (2017) suggest a framework based on four design principles to guide curriculum planning and teaching practice. It includes support sense-making (scaffolding tasks and amplify language usage so students can make their own meaning and optimizing output (strengthening the opportunities and supports for helping students to describe clearly their mathematical thinking to others, orally, visually, and in writing.

How *Math Expressions* Delivers

In Math Expressions, instructional approaches that foster mathematical habits of mind are integrated throughout. Using sense-making, math drawing, math structure, and math explaining, as well as real-world contexts, the program engages students in practices that grow and deepen math thinking habits. As indicated, research shows that when students use mathematical language and drawings to explain their thinking, they build conceptual understanding, make sense of quantities and their relationships in problem situations, make generalizations, and reason abstractly. With this major structure as part of its base, Math Expressions supports concept building and the mathematical approaches of the Mathematical Processes and Practices.

Math talk is an essential component of mathematical thinking. The frequent use of math talk explaining in Math Expressions lessons lifts the students to engage in mathematical habits of mind. During whole class or small group Math Talk, teachers orchestrate collaborative instructional conversations focused on the mathematical thinking of students, using responsive means of assistance that facilitate learning and teaching by all. During these exchanges, Student Leaders help others engage in mathematical thinking in a variety of ways, including coaching, modeling, clarifying, explaining, questioning, and offering feedback are just some of the ways these leaders collaborate with their classmates. Throughout Math Expressions Teacher Editions, embedded support at point-of-use is provided to assist teachers in creating the optimal Math Talk Learning Community experience and foster ongoing, generative, and collaborative mathematical thinking.

Learning Community

MathTalk **Best Practices** How did your students explain their math thinking today? Students are often unfamiliar with this process; they are accustomed to providing math answers only. Encouraging students to talk more fully about their thinking will take repeated efforts on your part. Expect this to be a building process that lasts for several weeks.

The benefits of Math Expressions' Math Talk Learning Community environment are multiple, particularly with regards to mathematical thinking. Describing one’s problem solving methods to another person can clarify one’s own thinking as well as clarify the matter for others. Another person’s approach can supply a new perspective, and frequent exposure to different approaches tends to engender flexible thinking. Math Talk creates opportunities to understand errors and permits teachers to assess students’ understanding on an ongoing basis. It encourages students to develop their language skills, both in math and in everyday English. Finally, Math Talk enables students to become active helpers and questioners, creating student-to-student talk that stimulates engagement and community.

Also key to how Math Expressions cultivates and facilitates mathematical thinking is the MathBoards program feature. Students use MathBoards as a tool for recording their thinking, relating their drawings to math language, and justifying their reasoning. Math Drawings focus on the mathematical aspects of a situation or of quantities. They may be generated by the student or teacher, or they may be research-based forms introduced in Math Expressions to help students understand and represent mathematical concepts, situations, and notation. Correct mathematical notation and vocabulary are always used in solutions and explanations, and these are related to the math drawings. But students also can use their own meaningful language to clarify concepts and make them memorable.

Problem Solving

“[S]olving a problem means finding a way out of a difficulty, a way around an obstacle, attaining an aim which is not immediately attainable” (Polya, 1965, p. ix). Problem-solving is an engrained and essential process within human experience. In the discipline of mathematics specifically, activities such as formulating problems and assessing the reasonableness of various approaches to their solutions is central to the development of skills and knowledge (Santos-Trigo, 2020). Engaging students in problem-solving tasks allows them to actively construct mathematical understandings and more deeply and with greater meaning than when teachers present information to students and have them carry out procedural exercises (Marcus & Fey, 2003; Masingila et al., 2018).

High-quality, research-based instructional math programs build on students’ intuitive mathematical thinking and unique background knowledge; incorporate rich and rigorous problem-solving tasks that engage interest; require that students employ strategic thinking and mathematical habits of mind—all with the larger aim of developing, over time, students' conceptual understanding and procedural fluency (Carpenter, Fennema, Franke, Levi, & Empson, 2015; David & Greene, 2007; Hiebert et al., 1996; NCTM, 2014). When students are authentically engaged in doing math, they are not simply solving problems; the problems themselves are opportunities for developing mathematical practices and habits of mind (Shoenfeld, 2020). "Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies" (NCTM, 2014, p. 17). "Math talk" is also an essential component of mathematical thinking (Cuoco et al. 1996).

Task selection is a critical aspect of supporting elementary students' reasoning and understanding in mathematics—and among the key features of effective instructional tasks are that they be challenging and connective and as well as open to multiple representations and multiple strategies for solutions (Childs & Glenn-White, 2018; Francisco & Maher, 2005; Mueller, Yankelewitz, & Maher, 2014). Tasks that consistently encourage high-level student thinking and reasoning (versus those that are routinely procedural) yield the greatest learning; and tasks of higher cognitive demand are necessary when promoting reasoning and problem solving in the mathematics classroom (Miri, David, & Uri, 2007; NCTM, 2014). In constructing mathematical tasks, it is further recommended that teachers “problematize with the goal of understanding the situations and developing solution methods that make sense” (Hiebert et al., 1996, p. 19). Mathematical habits of mind develop as a by-product of teaching mathematics through problem solving, in a process that entails modeling and reflection so that habits are internalized (Levasseur & Cuoco, 2009). Ultimately, problem solving in the mathematics classroom encourages students to see that their actions can lead to intellectual growth, and this “focus on the potential of students to develop their intellectual capacity provides a host of motivational benefits” (Blackwell et al., 2007, p. 260).

Students learn best when what they learn is relevant and meaningful. Connecting problem-solving tasks to real-world contexts and applications improves perceptions of the content as interesting and beneficial, thereby increasing motivation to learn (Fosnot & Dolk, 2010; Verschaffel, Depaepe, & Van Dooren., 2020). Students at all levels need to connect the mathematics they are learning to the world around them (Alberti, 2013), but research shows this to be particularly necessary for students historically underrepresented within STEM education and careers (Jackson, Mohr-Schroeder, Bush, Maiorca, Roberts, Yost, & Fowler, 2021; Wieselmann, Roehrig, & Kim, 2020) and for advanced learners (VanTassel-Baska & Brown, 2007). “When instruction is anchored in the context of each learner’s world, students are more likely to take ownership for…their own learning” (McREL, 2010, p. 7).

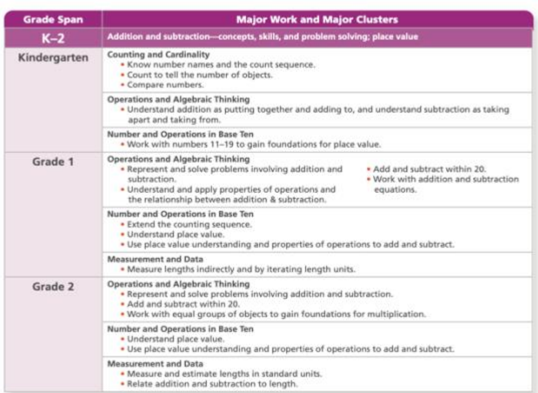
Making connections between new information and students’ existing knowledge— knowledge of other content areas and of the real world— has proved to be more effective than learning facts in isolation (Bransford et al., 1999; Caine & Caine, 1991; Hammond, 2014; Williams & Lombrozo, 2013; Willingham, 2006). Engaging students in problem-solving tasks allows them to actively construct mathematical understandings more deeply and with greater meaning than when teachers present information to students and have them carry out procedural exercises (Masingila et al., 2018). In their study of mathematics learning in early childhood, Cross and colleagues (2009) concluded that to effectively foster students’ conceptual understanding, teachers must include four key elements or opportunities within their teaching and learning activities: analyzing and reasoning; creating; integrating; and making real-world connection. “Our findings suggest that if teachers purposefully and persistently practice higher order thinking strategies for example dealing in class with real-world problems, encouraging open-ended class discussions, and fostering inquiry-oriented experiments, there is a good chance for a consequent development of critical thinking capabilities” (Miri et al., 2007, p. 353).

Mueller and colleagues (2014) studied specific teacher actions that encouraged students to take responsibility for their mathematical problem solving and assume roles that might otherwise be expected as the teacher’s responsibility, such as determining if solutions to a problem are correct, evaluating the reasonableness of arguments, and posing questions. "Rather than correcting students’ errors, the teachers charged the students with considering the reasonableness of solutions. Students were not praised for correct solutions; rather, all solutions were considered and students were afforded the opportunity to defend and/or modify their arguments. A result was that the learners were comfortable judging their own solutions and those of their peers, and learned that they could determine the validity of a mathematical argument" (p. 16-17).

In the elementary grades, students must develop understanding and make use of the "big ideas" in mathematics and problem-solving tasks in ways that also contribute to understanding of those big ideas. These big ideas constitute overarching concepts that connect multiple concepts, procedures, or problems within or across domains or topics. They also serve as an important aspect of the process of forming connections and acquiring background knowledge that can be applied to expand later understanding (Charles, 2005; Cross et al., 2009; NCTM, 2014).

How Math Expressions Delivers

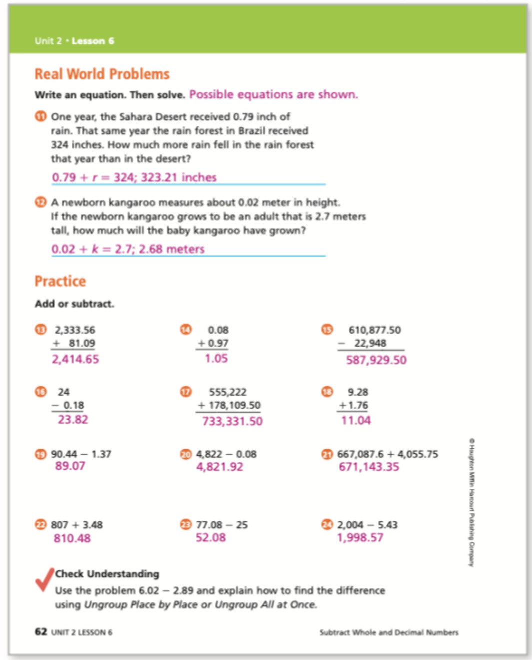
Within the *Math Expressions* curriculum, a series of learning progressions reflect research on students' natural stages when mastering concepts such as computation and problem-solving strategies. These learning stages informed the order of concepts, the sequence of units, and the positioning of topics in *Math Expressions*. The **Inquiry Learning Path** is paved with meaningful, collaborative problem solving throughout its four phases of **introduction of concepts, unfolding, kneading of knowledge, and maintaining and integrating fluency**. The emphasis on problem solving is reflected across the **Major Work** at every grade level of *Math Expressions*.



Grade Span	Major Work and Major Clusters
K–2	Addition and subtraction—concepts, skills, and problem solving; place value
Kindergarten	**Counting and Cardinality** • Know number names and the count sequence. • Count to tell the number of objects. • Compare numbers.
	Operations and Algebraic Thinking • Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.
	Number and Operations in Base Ten • Work with numbers 11–19 to gain foundations for place value.
Grade 1	**Operations and Algebraic Thinking** • Represent and solve problems involving addition and subtraction. • Understand and apply properties of operations and the relationship between addition & subtraction. • Add and subtract within 20. • Work with addition and subtraction equations.
	Number and Operations in Base Ten • Extend the counting sequence. • Understand place value. • Use place value understanding and properties of operations to add and subtract.
	Measurement and Data • Measure lengths indirectly and by iterating length units.
Grade 2	**Operations and Algebraic Thinking** • Represent and solve problems involving addition and subtraction. • Add and subtract within 20. • Work with equal groups of objects to gain foundations for multiplication.
	Number and Operations in Base Ten • Understand place value. • Use place value understanding and properties of operations to add and subtract.
	Measurement and Data • Measure and estimate lengths in standard units. • Relate addition and subtraction to length.

The sustained focus on problem solving throughout the curriculum is contextualized within and made more relevant via ongoing sourcing from and emphasis on real-world situations. This is interwoven throughout the program's **Big Idea** organizational structure of lessons and student activities as well as in embedded teacher support to guide and enrich Math Talk learning experiences.

Student Activity Book page 62



Unit 2 • **Lesson 6**

Real World Problems

Write an equation. Then solve. Possible equations are shown.

11 One year, the Sahara Desert received 0.79 inch of rain. That same year the rain forest in Brazil received 324 inches. How much more rain fell in the rain forest that year than in the desert?

$0.79 + r = 324$; 323.21 inches

12 A newborn kangaroo measures about 0.02 meter in height. If the newborn kangaroo grows to be an adult that is 2.7 meters tall, how much will the baby kangaroo have grown?

$0.02 + k = 2.7$; 2.68 meters

Practice

Add or subtract.

13 $2{,}333.56 + 81.09$ = 2,414.65

14 $0.08 + 0.97$ = 1.05

15 $610{,}877.50 - 22{,}948$ = 587,929.50

16 $24 - 0.18$ = 23.82

17 $555{,}222 + 178{,}109.50$ = 733,331.50

18 $9.28 + 1.76$ = 11.04

19 $90.44 - 1.37$ 89.07

20 $4{,}822 - 0.08$ 4,821.92

21 $667{,}087.6 + 4{,}055.75$ 671,143.35

22 $807 + 3.48$ 810.48

23 $77.08 - 25$ 52.08

24 $2{,}004 - 5.43$ 1,998.57

Check Understanding
Use the problem $6.02 - 2.89$ and explain how to find the difference using *Ungroup Place by Place or Ungroup All at Once.*

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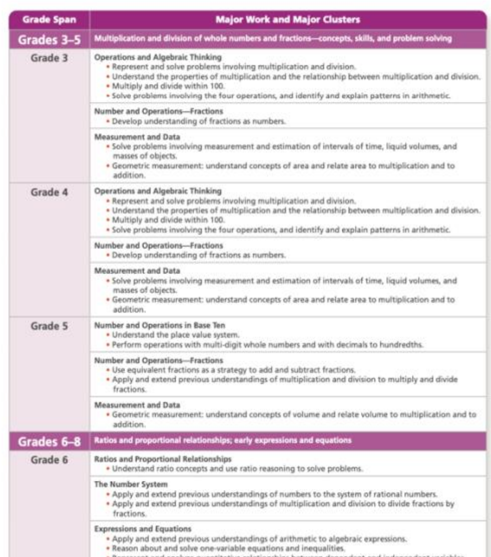
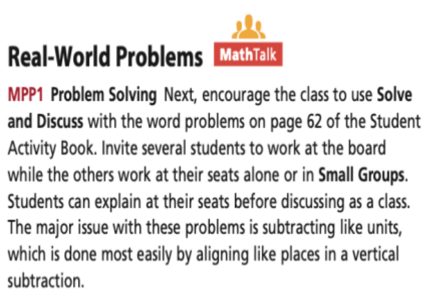
62 UNIT 2 LESSON 6 Subtract Whole and Decimal Numbers



Grade Span	Major Work and Major Clusters
Grades 3–5	Multiplication and division of whole numbers and fractions—concepts, skills, and problem solving
Grade 3	**Operations and Algebraic Thinking** • Represent and solve problems involving multiplication and division. • Understand the properties of multiplication and the relationship between multiplication and division. • Multiply and divide within 100. • Solve problems involving the four operations, and identify and explain patterns in arithmetic.
	Number and Operations—Fractions • Develop understanding of fractions as numbers.
	Measurement and Data • Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects. • Geometric measurement: understand concepts of area and relate area to multiplication and to addition.
Grade 4	**Operations and Algebraic Thinking** • Represent and solve problems involving multiplication and division. • Understand the properties of multiplication and the relationship between multiplication and division. • Multiply and divide within 100. • Solve problems involving the four operations, and identify and explain patterns in arithmetic.
	Number and Operations—Fractions • Develop understanding of fractions as numbers.
	Measurement and Data • Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects. • Geometric measurement: understand concepts of area and relate area to multiplication and to addition.
Grade 5	**Number and Operations in Base Ten** • Understand the place value system. • Perform operations with multi-digit whole numbers and with decimals to hundredths.
	Number and Operations—Fractions • Use equivalent fractions as a strategy to add and subtract fractions. • Apply and extend previous understandings of multiplication and division to multiply and divide fractions.
	Measurement and Data • Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition.
Grades 6–8	Ratios and proportional relationships; early expressions and equations
Grade 6	**Ratios and Proportional Relationships** • Understand ratio concepts and use ratio reasoning to solve problems.
	The Number System • Apply and extend previous understandings of numbers to the system of rational numbers. • Apply and extend previous understandings of multiplication and division to divide fractions by fractions.
	Expressions and Equations • Apply and extend previous understandings of arithmetic to algebraic expressions. • Reason about and solve one-variable equations and inequalities. • Represent and analyze quantitative relationships between dependent and independent variables.

Math Talk creates an inquiry environment that encourages constructive discussion of problem-solving methods through well-defined classroom activity structures. *Math Expressions* takes a research-based problem-solving approach, in which **students interpret the problem; solve the problem; represent the situation; and heck that the answer makes sense**. A significant part of the collaborative classroom culture in *Math Expressions* is the frequent exchange of problem-solving strategies during Math Talks. **Solving problems in this way the mathematics classroom has numerous benefits** for students because they integrate their conceptual understandings with procedural fluency; develop more positive views of their abilities to solve problems; demonstrate and build persistence; and view the discipline of mathematics more positively.



Real-World Problems MathTalk

MPP1 Problem Solving Next, encourage the class to use **Solve and Discuss** with the word problems on page 62 of the Student Activity Book. Invite several students to work at the board while the others work at their seats alone or in **Small Groups**. Students can explain at their seats before discussing as a class. The major issue with these problems is subtracting like units, which is done most easily by aligning like places in a vertical subtraction.

Matific's integration with *Math Expressions* enhances learning by empowering students to explore math with creativity, independence, and community. Aligning with the **Inquiry Path to Learning** through boosting problem-solving skills, **Matific** boosts students' **critical thinking skills** and adds **real-world relevance**, presenting problems in natural scenarios to increase engagement.

Productive Struggle

"An effective teacher provides students with appropriate challenges, encourages perseverance in solving problems, and supports productive struggle in learning mathematics" (NCTM, 2014, p. 11). Through productive perseverance, students grapple with the issues and are able to find solutions on their own, allowing them to persist and build resilience as they pursue learning and understanding (Jackson & Lambert, 2010), realizing that through effort and tenacity alongside sense making and problem solving, they are capable of doing well in mathematics (NCTM, 2014).

Research shows that "productive struggle" is necessary to the process of learning mathematics with understanding; when students are given opportunity to grapple with ideas, make mistakes, persist through difficulties, and arrive at solutions, learning outcomes improve (Kapur, 2014; Warshauer, 2015). It has also been found that students given time to make mistakes and persist through their struggles ultimately show greater understanding on posttest measures than their counterparts (Kapur, 2010). Perseverance through problem-solving encourages students to think about their own thinking and to discover that learning can happen without rushing to simply find the correct answer (Hiebert & Grouws, 2007). "Developing a productive disposition requires frequent opportunities to make sense of mathematics, to recognize the benefits of perseverance, and to experience the rewards of sense making in mathematics" (NRC, 2001, p. 131).

To effectively foster students' productive struggles and dispositions and thereby also mathematical habits of mind, teachers must continually demonstrate, through modeling, instructional activities, and classroom culture, how challenge is a natural part of the learning process (Mason, 2020). Teachers must allow students to experience the benefits of perseverance and provide specific descriptive feedback to students on their progress related to their efforts (Hattie & Clarke, 2018; Hattie & Timperley, 2007; Star, 2015). Other practices that support productive perseverance include heterogeneous grouping, effective teacher-directed questioning, setting problems in a setting familiar to students and that draws from their everyday lives, plus goal setting before and reflection after problem-solving (Pasquale, 2016). Cultivating students' productive dispositions also requires that teachers carefully select tasks and provide reassurance and guidance that students need to complete the tasks—but without diminishing the cognitive demand of the task or giving students too much help or direct answers. Students need sufficient time, not only to persist through challenging and devise solutions, but also to develop curiosity and stamina (Goldenberg, et al., 2015; Pascale, 2016). The kinds of questions teachers ask and the kind of support that teachers offer are critical, as they either facilitate or undermine the productive efforts of students' struggles and determine whether students view struggle as a positive endeavor or the source of difficulty and frustration (Warshauer, 2015). Timing of support also plays a vital role. When scaffolding is given to students before they have the opportunity to make sense of a challenging task independently, they are inhibited in the process of developing productive perseverance. "All too often, so much support is provided through the initial scaffolding that the cognitive demand of the task is significantly decreased (Boston & Wilhelm, 2015). If this sort of scaffolding is provided upfront for students who struggle, then these same students are denied access to cognitively demanding tasks. When access is denied, equity becomes an issue" (Dixon, 2018, online).

Productive perseverance makes important contributions in the promotion of a growth mindset. A concept pioneered by psychologist Carol Dweck (2006 & 2008), growth mindset is a belief that a person's intelligence, competence, and talents can be developed through dedicated efforts and hard work. "To ensure that all students have access to an equitable mathematics program, educators need to identify, acknowledge, and discuss the mindsets and beliefs that they have about students' abilities" (NCTM, 2014, p. 64). A growth mindset within mathematics emphasizes teaching and learning as processes that cultivate mathematical abilities; stresses that success and learning are reflections of effort and not intelligence; and promotes a belief that all students are capable of participating and achieving in mathematics. Society has traditionally valued the math learner who can memorize well and calculate fast, rather than others who possess equal potential but may be deeper, slower, and possibly more creative. These earlier mindsets have contributed to persistent negative perceptions within mathematics education specifically—and the evolution toward pervasive growth mindsets are necessary if all students are to be successful math learners. Teachers should foster and display a growth mindset by valuing all students' thinking and efforts while also relying on pedagogical practices such as differentiated tasks, mixed-ability groupings, and praise for students' contributions and perseverance within their mathematical learning (Dweck 2006 & 2008; NCTM, 2014). Schools and classrooms that reinforce growth mindset messaging make learning enjoyable and place the focus on that learning rather than on students' performance (Yeager et al., 2013).

Setting and supporting rigorous expectations and a genuine belief that student effort and effective instruction outweigh "smarts" and life circumstances increase students' opportunities to learn—and create a more equitable learning experience (NCTM, 2014). It is critical, then, to implement differentiated processes for instruction that foster students' mathematical thinking and broaden students' productive engagement with mathematics in ways that also support individual students as needed, meeting them at their developmental level with a positive, appropriate level of challenge (Muñiz, 2020).

How *Math Expressions* Delivers

From the first day of the school year to the last, *Math Expressions* builds and sustains a safe, supportive classroom culture in which students explicitly recognize and directly experience how struggle is a productive and necessary part of authentic learning. Students are allowed, even encouraged, to make mistakes and share the thinking behind them while guided collaboration among students yields constructive, collective resolutions and strategy development.

The program urges teachers to model such dispositions and provides ongoing opportunities for students to analyze and correct errors, explaining why the reasoning was flawed. The **Teacher Edition** includes myriad suggestions to help teachers watch for situations that may lead to common errors. As a part of the **Unit Review/Test Teacher Edition** pages, common errors and prescriptions are listed for selected test items.

Teaching Note

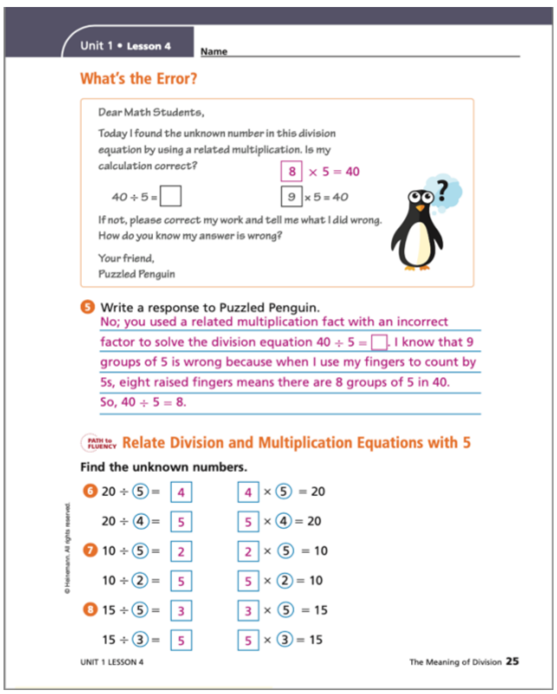
Learning Community

Mistakes Students love it when a teacher makes a mistake and they can help the teacher. This makes mistakes seem less bad. The teacher pretending to make a mistake is a game that enables the teacher to introduce and discuss errors with no student at fault. You can use this game all year to review and discuss errors. Students need to explain why an error is an error and why something is correct. We introduce this game today to relate to Puzzled Penguin. Students will help Puzzled Penguin all year.

Common Errors and Prescriptions for Selected Items

Items	Common Error	Prescription
4, 5	Adds numerators and denominators, subtracts numerators and denominators	Use a number line or fraction models to demonstrate for students why only the numerator should be added or subtracted when fractions have the same denominator.
6	Does not use least common denominator to add or subtract fractions with different denominators	For $\frac{3}{4} - \frac{1}{3}$, have students list all the multiples of 4 and 3, then circle the multiples that are common to both lists. Discuss why 12 is the least common multiple of 4 and 3.
7	Does not regroup when the sum includes a fraction greater than 1	Remind students that the fraction part must be less than 1. For example, for Item 7, $4\frac{2}{3} + 3\frac{2}{3} = 7\frac{4}{3} = 8\frac{1}{3}$. Explain that leaving an answer as $7\frac{4}{3}$ would not be acceptable.
8	May not think that fractions made up of greater numbers such as $\frac{21}{28}$ are equivalent to $\frac{3}{4}$	Remind students that they can multiply both the numerator and denominator in a fraction by the same number to find an equivalent fraction. Guide students through using this strategy to determine which fractions in the list are equal to $\frac{3}{4}$.
9	May not recognize fractions can be simplified to form an equivalent fraction.	Explain to students that when a fraction is simplified, the two fractions are equivalent. Ask students what numbers divide evenly into both the numerator and denominator for the fraction $\frac{8}{12}$. (2, and 4) Walk students through the process of dividing the numerator and denominator of $\frac{8}{12}$ by 2 or by 4, to find equivalent fractions $\frac{4}{6}$ and $\frac{2}{3}$. Use fraction strips or circles to prove equivalence.
1–3	Makes incorrect comparisons	Remind students that they can always compare fractions by rewriting them as equivalent fractions with a common denominator. For Item 2, walk students through the steps of finding a common denominator and then rewriting and comparing the fractions.
13, 14	Incorrectly keeps the same numerators of the original fraction when writing equivalent fractions.	Remind students that when they multiply or divide the denominator of a fraction by a number, they must also multiply or divide the numerator by the same number.
4–6, 14–16	Chooses the wrong operation	Suggest that students think of a simpler problem with whole numbers. Once they determine the correct operations, they can go back to using the fractions or mixed numbers in the problem.

To support a learning community in which productive struggle is normalized and celebrated, *Math Expressions* features a recurring character and accompanying activity across the curriculum: **Puzzled Penguin**. Puzzled Penguin shows students typical errors learners make in grappling with specific math concepts and skills. Common errors are presented in **Puzzled Penguin letters to students** throughout the program in conjunction with target lessons. Students enjoy teaching Puzzled Penguin the correct way, why this way is correct, and why Puzzled Penguin made the error.



Unit 1 • Lesson 4 Name

What's the Error?

Dear Math Students,

Today I found the unknown number in this division equation by using a related multiplication. Is my calculation correct?

$40 \div 5 = \boxed{}$ $\boxed{9} \times 5 = 40$ $\boxed{8} \times 5 = 40$

If not, please correct my work and tell me what I did wrong. How do you know my answer is wrong?

Your friend,
Puzzled Penguin

5 Write a response to Puzzled Penguin.
No; you used a related multiplication fact with an incorrect factor to solve the division equation $40 \div 5 = \boxed{}$. I know that 9 groups of 5 is wrong because when I use my fingers to count by 5s, eight raised fingers means there are 8 groups of 5 in 40. So, $40 \div 5 = 8$.

PATH to FLUENCY Relate Division and Multiplication Equations with 5

Find the unknown numbers.

6 $20 \div 5 = \boxed{4}$ $\boxed{4} \times 5 = 20$

$20 \div 4 = \boxed{5}$ $\boxed{5} \times 4 = 20$

7 $10 \div 5 = \boxed{2}$ $\boxed{2} \times 5 = 10$

$10 \div 2 = \boxed{5}$ $\boxed{5} \times 2 = 10$

8 $15 \div 5 = \boxed{3}$ $\boxed{3} \times 5 = 15$

$15 \div 3 = \boxed{5}$ $\boxed{5} \times 3 = 15$

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UNIT 1 LESSON 4 The Meaning of Division 25

Math Expressions' seamless integration with ***Matific*** provides **personalized and adaptive learning pathways** that allow for **optimal levels of challenge** for each and every student along with **differentiated practice** for students struggling with specific concepts, addressing individual learning gaps. Cutting-edge learning technology empowers teachers with embedded assessment features to quickly evaluate students' problem-solving skills and critical thinking abilities. Easy to assign digital and printable activities provide teachers real-time reporting and formative data that helps identify areas where students may be struggling. This enables teachers to make data-informed instructional decisions that drive student progress across both platforms and allow for a comprehensive understanding of their learning trajectory. *Matific's* activities focus on developing conceptual understanding alongside procedural fluency, reinforcing the *Math Expressions* curriculum. Such close alignment with *Matific* advances *Math Expressions* highly structured capacity to create a cohesive, streamlined learning experience for all students.

Assessment in Support of Learning

"An excellent mathematics program ensures that assessment is an integral part of instruction, provides evidence of proficiency with important mathematics content and practices, includes a variety of strategies and data sources, and informs feedback to students, instructional decisions, and program improvement" (NCTM, 2014, p. 5). Data-driven instructional decision-making is the systematic collection, analysis, and application of many forms of data from multiple sources in order to identify students' strengths and weaknesses regarding learning objectives and subsequently address student learning needs and optimize performance in future instruction. Rigorous, ongoing assessment that yields meaningful data is a fundamental component with an effective data-driven decision-making system, as is summative and performance assessment. Research indicates that when it is implemented well, data-driven instruction has the potential to dramatically improve student achievement (Bambrick-Santoyo, 2014; Dunn, Airola, Lo, & Garrison, 2013; Marsh, Pane, & Hamilton, 2006; Schifter, Natarajan, Ketelhut, & Kirchgessner, 2014). "Assessments should drive excellent teaching and ensure that all students learn at high levels. To do so, education policy and practice must encompass a broader range of assessments so that schools have complete and effective assessment systems. This system would include predictive, informative, and evaluative assessments based on the state’s standards and curriculum" (Jimenez & Modaffari, 2021, p. 18).

Key aspects of effective mathematics assessment practices include a focus on evidence that identifies indicators of students' mathematical thinking along learning progressions that show how their mathematical thinking develops over time (Fonger, Stephens, Blanton, Isler, Knuth, & Gardiner, 2018; NCTM, 2014; Thanheiser & Melhuish, 2023) as well as reliance on ongoing, integrated formative assessment to monitor progress and guide instructional decision-making. Effective assessment tools allow teachers to collect data about what is working and what is not so that they can take precise, swift, and effective action to better serve students (Black & Wiliam, 1998a, 1998b; Fuson et al., 2005; Hattie & Clarke, 2018; Hattie et al., 2017; Murata, 2013; NCTM, 2014). Just as we want students to be agents in their own learning, they must be active participants in their own assessment (Frey, Hattie, & Fisher, 2018; Hattie & Clarke, 2018; Heritage, 2013).

Math Expressions ©2026 offers a comprehensive assessment program, with tools provided for each stage of learning (diagnostic, formative, summative) and in varied formats. This includes ongoing, formative feedback generated through Math Talk discourse, which can be used to effectively guide students through Class Learning Pathways. Throughout, teachers have the information they need to effectively plan and modify instruction to support all students in attaining high levels of learning.

Monitoring Student Progress and Providing Ongoing Feedback

Research continually demonstrates that data-driven approaches to mathematics instruction, entailing comprehensive, ongoing monitoring of student progress toward meeting standards and other goals for learning and using that information to guide instruction, is key to developing students' math skills. A wealth of studies indicates that regular use of ongoing, flexible, integrated, and varied assessment to monitor student progress can mitigate and prevent mathematical weaknesses and improve student learning outcomes (Baker, Gersten, & Lee, 2002; Black & Wiliam, 1998a, 1998b, Clarke & Shinn, 2004; Hattie, 2009, 2012; Klute, Apthorp, Harlacher, & Reale, 2017; Lee, Chung, Zhang, Abedi, & Warschauer, 2020; Popham, 2008; Stecker, Fuchs, & Fuchs, 2005; Wiliam, 2011). “Using multiple types of assessments provides more insight into students’ learning because students have more than one way to demonstrate their knowledge and skills” (McREL, 2010, p. 44).

Assessment is an essential component of effective instruction and a process by which teachers can continuously monitor student understanding and progress toward meeting learning goals. “[T]eachers using assessment for learning continually look for ways in which they can generate evidence of student learning, and they use this evidence to adapt their instruction to better meet their students' learning needs" (Leahy, Lyon, Thompson, & Wiliam 2005, p. 23). Effective assessment tools allow teachers to collect data about what is working and what is not so that they can take precise, swift, and effective action to better serve students. "Assessment should not merely be done *to* students; it should also be done *for* students, to guide and enhance their learning (NCTM, 2000, p. 22). Teachers should collect a variety of variety of evidence before, during, and after instruction to evaluate progress and adjust instruction with the goal of best supporting each student. It is also critical that a broad range of measures and tasks be utilized diagnostically, formatively, and summatively to compile a comprehensive picture of a student's growth and track that growth over time (Krebs, 2005; NCTM, 2000 & 2014).

To make effective decisions about students' instructional needs, teachers rely on diagnostic assessment. Tailoring instruction and supplemental practice based on the results of valid diagnostic assessment improves learning outcomes (Mayes, Chase, & Walker, 2008). Diagnostic assessments provide data about students' prior knowledge and current skill levels within a domain as well as preconceptions or misunderstandings regarding learning material (Ketterlin-Geller & Yovanoff, 2009). A screening tool at the opening of the school year can help identify those who are at-risk or need additional support (Fuchs & Fuchs, 2006).

Research has long established that formative assessment is integral to an effective mathematics program (Black & Wiliam, 1998a, 1998b); NCTM (2020) calls for formative assessment to undergird and support learning interventions for individual students and promote equity. The phrase “formative assessment” encompasses the wide variety of activities—formal (such as quizzes or homework assignments) and informal (such as discussion and observation)—that teachers employ throughout the learning process to gather this kind of instructional data to assess student understanding and to make and adapt instructional decisions; formative assessment moves testing from the end into the middle of instruction, to guide teaching and learning as it occurs (Hattie & Clarke, 2018; Hattie & Timperley, 2007; Heritage, 2013; Jimenez & Modaffari, 2021). Reviews of studies examining formative assessment concluded that the "use of formative assessments benefited students at all ability levels" (National Mathematics Advisory Panel, 2008 p. 46). In a study of curriculum-based measurement, when teachers administered outcomes-based assessments regularly to monitor student progress and used data to make appropriate adjustments to instruction, students showed significant gains (Stecker et al., 2005). However, studies show that formative assessment is especially beneficial for lower-performing and at-risk students, including those historically underserved due to ethnicity, poverty, and disabilities and those enrolled in special education programs; monitoring student progress fairly and accurately through means that promote student awareness and agency and modifying instruction for individual students based on assessment data shrinks achievement gaps and improves overall achievement (Black & Wiliam, 1998a & 1998b; NCTM, 2015; 2020; Xenofontos, 2019).

Ongoing, formative assessment is essential for teachers to monitor and develop students' procedural fluency skills—and assessment type matters greatly. Timed tests commonly used to evaluate fluency have shown to be detrimental to children's fluency skills and likely to induce math anxiety; instead, interviews, observation, and journals are more effective and supportive in determining students' procedural fluency levels (Kling & Bay-Williams, 2014). Additionally, in the course of formative assessment, teachers must make constructive use of students' incorrect answers and, rather than simply mark them wrong, examine incorrect responses to see if they reveal specific misunderstandings (Popham, 2006). By analyzing student errors, teachers can determine which specific concepts, algorithms, or procedures need additional instruction (Ketterlin-Geller & Yovanoff, 2009). Further, discussing responses with students and authentically seeking to discover the thinking behind them within a safe, comfortable environment opens opportunities for deeper learning (Fuson et al., 2005). Critical components of effective formative classroom assessment include prompt feedback—both teacher to student and peer to peer—as well as the direct, active involvement of students themselves throughout the assessment process, which includes setting goals for learning and outcomes (Fuson et al., 2005; Hattie et al., 2018; Hattie & Timperley, 2007; Heritage, 2013; Kozleski, 2010; NCTM, 2020; NRC, 2012). Timely feedback from regular math assessment helps students evaluate their strengths and weaknesses and gauge their progress toward meeting clearly specified learning goals and helps teachers produce significant—and often substantial—gains in learning and performance (Black & Wiliam 1998a & 1998b; Cotton, 1995; Hattie, 2009 & 2012; Roschelle, Feng, Murphy, & Mason, 2016). "During the multiple opportunities for learning and engagement, teachers need to provide feedback to refine the student’s understanding of the content. Teachers need to plan for students’ misconceptions to be identified, explored and challenged, to make transparent the links with their prior experiences and to provide multiple opportunities and scaffolding to make those links with new information: the essence of effective feedback" (Hattie & Clarke, 2018, p. 3).

Math Talk learning communities facilitate consistent, collaborative progress monitoring and formative assessment. Because thinking is made visible and teachers and students alike provide and receive feedback, this yields ongoing indication of where learning is in relation to goals, which in turn allows for clarification, advancement of understanding and any needed adjustment to instruction on an individual and just-in-time basis (Fuson et al., 2005; Hattie et al., 2017; Hufferd-Ackles et al., 2004, 2015; Humphreys & Parker, 2015; Parrish, 2010; Parrish & Dominick, 2016).

How *Math Expressions* Delivers

Math Expressions offers educators continuous, flexible, and varied opportunities to monitor student progress and provide ongoing feedback to students, supporting them meet learning goals. Invaluable formative assessment happens organically and constantly within the Math Talk Learning Community via verbal, drawn, and written exchanges between teachers and students and while teachers attend to and facilitate students' math thinking and problem solving with each other. The production of such indication and evidence allow for evaluation of understanding and prompt intervention.

Differentiated Instruction

Universal Access
Some students have difficulty completing an assignment correctly. Give verbal feedback that involves first giving praise for what is correct and then giving suggestions for what could be improved. Give students an opportunity to correct the assignment. Be sure that the students have at least one task that they can complete successfully.

The *Math Expressions* Teacher Editions also guide and remind teachers in how to effectively provide feedback, including within the Responsive Means of Assistance model, which entails:

- engage and involve
- manage
- coach, model, clarify, explain, question, and give feedback

Quick Practice activities focus on vitally important skills and concepts that can be carried out in a whole-class format with immediate feedback. Additionally, teachers who use *Math Expressions* report that Quick Practice time provides an engaging start for math class, helps build a Helping Community, and enables all students to act as Student Leaders eventually—which combined serve to create a safe and supportive learning environment where progress can be monitored and feedback provided continually and constructively.

Quick Practice 5m

(See page QP1-U1.)
- **Multiplication Equations as Groups of 5** (A)
- **Count-Bys for 5: Look, Don't Look, Alternate** (B)
- **Find Product by Counting by 5** (C)
- **Find Unknown Factor (Divide) by Counting by 5** (D)

The *Math Expressions* comprehensive assessment system yields feedback to all and permits realistic adjustments of proximal learning goals by students and by the teacher. Beyond the Math Talk exchanges, the program offers a diagnostic, formative, and summative assessment system and intervention resources, which are described in the following section. Still more, the seamless alignment between *Math Expressions* and *Matific* offers teachers further embedded assessment solutions to readily, routinely evaluate students' problem-solving skills and critical thinking abilities. *Matific* additionally yields real-time reporting and formative data to help teachers identify areas where students may be struggling and make prompt data-informed decisions to drive student progress on an individual learner level. *Matific* adapts to individual student needs, adjusting difficulty and providing targeted practice based on performance to create personalized learning paths while teachers gain immediate insights into student understanding through *Matific's* data analysis tools and progress tracking.

A Comprehensive Assessment System

Assessment across a wide range of formats, timelines, and data points is fundamental to successful mathematics teaching and learning. A complete, effective assessment system is comprised of measures with one of three distinct purposes: to predict student performance, to inform instruction, or to evaluate learning; each measure must be in the service of high-quality instruction and optimal outcomes for all students (Jimenez & Modaffari, 2021). "The results of large-scale mathematics assessments should not be used as the sole source of information to make high-stakes decisions about schools, teachers, and students. High-stakes decisions should also take into account relevant and valid data on classroom-based performance, such as formative and summative assessments of high quality that offer students a range of opportunities to demonstrate their mathematical knowledge. Moreover, educational systems—states, districts, and schools—should be held accountable for providing essential support for high-quality mathematics teaching and learning before teachers and students are held accountable for assessment results" (NCTM, 2016, online).

Summative assessment differs from those that are formative or diagnostic in nature because the purpose of summative assessment is to determine the student's overall achievement in a specific area of learning at a particular time. Teachers can effectively use summative assessments as another measure, in another point in time, and with another means by which to best evaluate student understanding. As part of an integrated assessment system, summative measures can also help teachers shape instruction and differentiate to personalize learning. Summative assessments are also useful as accountability measures for grading and gauging student learning against a set of standards or expectations. Summative assessments provide evaluative information to teachers about the effectiveness of their instructional program. Research indicates that classroom summative assessments also have the potential to positively impact learning (Harlen, 2005; Moss, 2013; NCTM, 2016). While traditionally summative assessment has been associated with higher stakes testing, there is a role for summative assessment in the classroom when used as an additional constructive measure demonstrating progress at a particular point of time. Evaluating student learning at the end of a unit or chapter provides insight when used as a point of information to guide subsequent instruction (Black & Wiliam, 1998a & 1998b).

Performance tasks allow teachers to engage students in real-world activities and model "what is important to teach and…what is important to learn" (Lane, 2013, p. 313). Assessment systems in high-performing nations "emphasize deep knowledge of core concepts within and across the disciplines, problem-solving, collaboration, analysis, synthesis, and critical thinking. As a large and increasing part of their examination systems, high-achieving nations use open-ended performance tasks…to give students opportunities to develop and demonstrate higher order thinking skills" (Darling-Hammond, 2010, p. 3). Performance-based tasks may take different forms, require different types of performances, and be used for different purposes (formative or summative), but they are typically couched in an authentic or real-life scenario and require high-level thinking. Research has established the benefits of performance-based assessment. A review of classroom assessment practices in an age of high-stakes testing led Schneider, Egan, and Julian (2013) to conclude that “the value of high-quality performance tasks should not be diminished and should be encouraged as an important tool” (p. 66). Performance-based assessments in STEM subjects specifically provide an alternative and complement to standardized achievement tests because they enable a holistic evaluation of the performance of an individual student. For reasons of equity and accurate representation of individual students' knowledge and skills, performance-based assessment appropriate for students from low SES levels are essential (Zimmerman, Maker, & Alfaiz, 2020).

Well-designed assessment conducted regularly and used by teachers to alter and improve instruction can have tremendous impact on students’ learning (NRC, 2005). “We believe that assessment, whether externally mandated or developed by the teacher, should support the development of students’ mathematical proficiency. It needs to provide opportunities for students to learn rather than taking time away from their learning” (NRC, 2001, p. 423). “When viewed as a process that is indistinguishable from effective instruction, assessment serves as a means to achieve productive teaching and learning for all, rather than merely as the final stage in the traditional teach-learn-assess cycle” (NCTM, 2014, p. 94).

How *Math Expressions* Delivers

Math Expressions provides abundant, frequent, and comprehensive **Assessment, Review, and Intervention Resources.** These range from **Diagnostic Tools** for both **Formative Assessment** and **Summative Assessment** as well as **Review** and **Intervention Resources** to support the learning needs of students.

Measurement and intervention across the *Math Expressions* curriculum take the following forms:

Diagnostic Tools	Formative Assessment	Review Resources
Student Activity Book • Unit Review/Test • Quick Quiz 1 • Quick Quiz 2 • Performance Task **Digital Assessment & Instruction** • Heinemann *Flight*	**Student Activity Book** • Check Understanding (in every lesson) **Teacher Edition** • Quick Practice (in every lesson) • Math Talk (in every lesson) • Portfolio Suggestions **Summative Assessment** **Assessment Guide** • Unit Assessment, Form A and B • Performance Assessment	**Student Activity Book** • Strategy Check 1 • Strategy Check 2 **Homework and Remembering** • Review of recently taught topics • Spiral Review **Teacher Edition** • Unit Review/Test **Digital Assessment & Instruction** • Heinemann *Flight* **Intervention Resources** • Response to Intervention • Interactive Math Practice

They are provided at the following intervals:

Daily	Big Idea	Unit
Check Understanding	Quick Quiz Strategy Check	Performance Task Performance Assessment Unit Review/Test Unit Assessment, Form A and B

Such assessment tools and resources are in addition to the ongoing, continuous opportunities for teachers to monitor progress and provide feedback within the **Math Talk Learning Community** as well as the assessment, reporting, and data-driven instructional decision features provided by the integration of the ***Matific*** adaptive learning platform within the *Math Expressions* curriculum, as described previously.

The Mastery Learning Loop

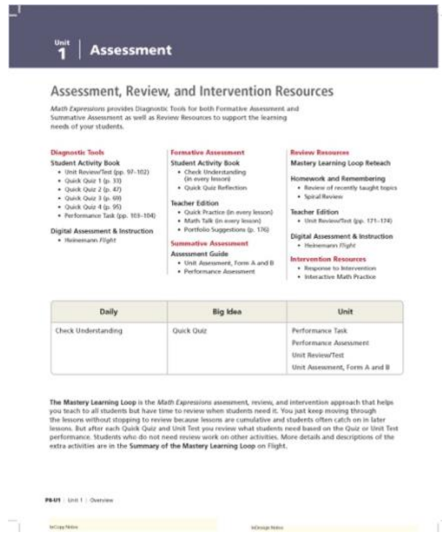
"[M]athematics instruction—like any good instruction—must be intentionally designed and carefully orchestrated in the classroom, and should always focus on impacting student learning" (Hattie et al., 2017, pp. 3–4). Examinations of teaching in American mathematics classrooms concurrent with standards reform efforts of the past few decades have shown a lack of depth and rigor, as well as diffused coverage of content (Fuson, 2000; NRC, 2001; Schmidt, 2012). In international comparisons of math and science performance, the countries at the top generally present students with fewer topics but at greater depth and increased coherence—and this contributes to persistent gaps in U.S. mathematics achievement. Without substantial and sustained changes to its educational system, specifically in mathematics, the United States will continue to slip when stacked against other nations, and the gap between the country's highest and lowest achievers will continue to grow (National Center for Education Statistics [NCES], 2021; National Mathematics Advisory Panel, 2008; Schmidt et al., 2005). Reviews of math curricula suggest that a greater focus on fewer core mathematical ideas at each grade yields a greater depth of understanding that results in higher levels of content mastery and greater potential for rigor (Achieve, 2010; Cobb & Jackson, 2011); therefore, teachers must make choices among wide-ranging standards based on the specific needs of their students to ensure sufficient time is available to teach concepts and procedures effectively (NCTM, 2014; Senn, Rutherford, & Marzano., 2014; Wiggins & McTighe, 2005). "The mathematics curriculum in Grades PreK–8 should be streamlined and should emphasize a well-defined set of the most critical topics in the early grades" (National Mathematics Advisory Panel, 2008, p. xiii). Elementary-level mathematics programs must build a strong foundation of deep mathematical understanding resulting from students' solid conceptual understanding with an emphasis on sense-making and reasoning and through a coherent, cohesive instructional approach at the early childhood and elementary level. “When mathematics instruction goes deep, children are empowered to explore the richness of the mathematical landscape" (NCTM, 2020, p. 77). By emphasizing critical thinking, problem-solving, and mathematical states of mind, educators urged to adopt an approach that prioritizes depth over breadth (Noguera, Darling-Hammond, Friedlaender, 2015). Additionally at K-6, students must develop understanding and make use of the "big ideas" in mathematics and problem-solving tasks in ways that also contribute to understanding of those big ideas as they advance in learning (Charles, 2005; Cross et al., 2009; NCTM, 2014). What and how students are taught should reflect the key ideas within an academic discipline that determine how knowledge is organized and generated within that discipline; a coherent set of content standards must evolve from particulars (e.g., the meaning and operations of whole numbers, including simple math facts and routine computational procedures associated with whole numbers and fractions) to deeper structures inherent in the discipline, with these deeper structures then serving as a means for connecting the particulars (such as an understanding of the rational number system and its properties (Schmidt, Houang, & Cogan, 2002). A cohesive, coherent math curriculum is sequenced within and across grade levels in a way that best reflects the hierarchical and logical structures of mathematics and to develop essential, deep understanding of concepts and procedures (NCTM, 2020; Schmidt et al., 2005).

Learning progressions are a “carefully sequenced set of building blocks that students must master en route to a more distant curricular aim. The building blocks consist of sub skills and bodies of enabling knowledge” (Popham, 2007, p. 83). Teachers should support learners as they build on what they know, develop more complex understandings, and realize that mathematics is not a set of discrete parts—it is coherent and connected (Fosnot & Dolk, 2010; Ma, 2010). "[L]earning progressions can be leveraged in mathematics education as a form of curriculum research that advances a linked understanding of students learning over time through careful articulation of a curricular framework and progression, instructional sequence, assessments, and levels of sophistication in student learning" (Fonger et al., 2018, abstract). Such progressions are ideally underpinned by clearly identified, research-based, foundational as well as achievable goals for instruction that form learning trajectories that relate and build understandings that aid all children advance their mathematical skills and knowledge; while such trajectories yield optimal learning outcomes for all students, some students, including those with special needs or from historically disadvantaged backgrounds, may require additional scaffolding or support on their individual pathway to mastery as well as to make their day-to-day experiences learning math challenging-but-achievable and their skills and knowledge (Clements, Fuson, & Sarama, 2017).

Murata and Fuson (2006, 2016, Murata, 2013) offer a Class Learning Zone within their Class Learning Path model that provides a mastery learning loop approach to effective intervention in diverse classrooms where one teacher must assist large groups of students, such as those in a typical public elementary school in the U.S. The model draws on the work of Tharp and Gallimore (1988) as well as Vygotsky (1978, 1986 - specifically Zones of Proximal Development (ZPD)) and takes a "responsive" approach to teaching. The model operates as follows. When introducing a new topic, the teacher, in the Class Learning Zone, orients students and elicits their initial knowledge as well as methods for solving problems or thinking about the context. With assistance from teaching materials as well as examples and explanations from students demonstrating understanding, the teacher then moves students along the Class Learning Path. Ongoing formative assessment may yield indication of exception for some students—either advanced or delayed—outside of the Class Learning Zone. "The Class Learning Path is the day-to-day sum of the learning paths of most of the students in the class (reflecting the state of growth in their methods and in their understanding each day), but this falls within manageable groups of related-enough mathematical assistance needs. A few students may not fully master a target solution method, but assistance will continue in subsequent units toward mastery or with additional help outside of class" (Murata & Fuson, 2016, p. 80). The researchers also note that their previous findings have shown that for many mathematics topics, some typical errors stem from partial and incomplete understandings while some more random errors occur due to lapses of attention or effort, and there are usually several solution methods that are limited in number and variable in their sophistication, generalizability, and ease of understanding. So, while there may be 25 students in a given classroom, there are not that many different strategies or learning paths; instead teachers typically may need to responsively assist with 5-6 incorrect or partial understandings, with minor variations. Such situations can be addressed via curricular materials and visual supports. Teachers can also coordinate individual students' learning paths related to specific topics so as to support and stretch each other's ZPDs (Murata, 2013). The mastery learning loop effectively differentiates mathematics instruction as called for in the work of Tomlinson (1997, 2001, 2017) and Lawrence-Brown (2004). The Mastery Learning Loop also fosters self-regulated learning, helping students, through metacognition, behavior, and motivation, more cognizant of and responsible for their own learning (Frey et al., 2018; Hattie, 2012; Labuhn, Zimmerman, & Hasselhorn, 2010).

How *Math Expressions* Delivers

The Mastery Learning Loop is the Math Expressions assessment, review, and intervention approach that helps you teach to all students but have time to review when students need it. You just keep moving through the lessons without stopping to review because lessons are cumulative and students often catch on in later lessons. But after each Quick Quiz and Unit Test you review what students need based on the Quiz or Unit Test performance. Students who do not need review work on other activities.



Unit 1 | Assessment

Assessment, Review, and Intervention Resources

Math Expressions provides Diagnostic Tools for both Formative Assessment and Summative Assessment as well as Review Resources to support the learning needs of your students.

Diagnostic Tools

Student Activity Book
- Unit Review/Test (pp. 97–102)
- Quick Quiz 1 (p. 33)
- Quick Quiz 2 (p. 47)
- Quick Quiz 3 (p. 69)
- Quick Quiz 4 (p. 95)
- Performance Task (pp. 103–104)

Digital Assessment & Instruction
- Heinemann *Flight*

Formative Assessment

Student Activity Book
- Check Understanding (in every lesson)
- Quick Quiz Reflection

Teacher Edition
- Quick Practice (in every lesson)
- Math Talk (in every lesson)
- Portfolio Suggestions (p. 176)

Summative Assessment

Assessment Guide
- Unit Assessment, Form A and B
- Performance Assessment

Review Resources

Mastery Learning Loop Reteach

Homework and Remembering
- Review of recently taught topics
- Spiral Review

Teacher Edition
- Unit Review/Test (pp. 171–174)

Digital Assessment & Instruction
- Heinemann *Flight*

Intervention Resources
- Response to Intervention
- Interactive Math Practice

Daily	Big Idea	Unit
Check Understanding	Quick Quiz	Performance Task Performance Assessment Unit Review/Test Unit Assessment, Form A and B

The Mastery Learning Loop is the *Math Expressions* assessment, review, and intervention approach that helps you teach to all students but have time to review when students need it. You just keep moving through the lessons without stopping to review because lessons are cumulative and students often catch on in later lessons. But after each Quick Quiz and Unit Test you review what students need based on the Quiz or Unit Test performance. Students who do not need review work on other activities. More details and descriptions of the extra activities are in the **Summary of the Mastery Learning Loop** on Flight.

P8-U1 | Unit 1 | Overview

Developing All Math Learners

A positive and valued mathematics identity at a young age is key to building a strong mathematical foundation (Clements & Sarama, 2020; NCTM, 2020). Math knowledge at the primary grades, particularly children's competence with quantity and number as well as geometric and spatial reasoning, predicts later school achievement not only in math, but also other subjects, including reading; in fact, early math knowledge is one of the strongest predictors of a student's math grades in high school and at high school graduation and college entry (Fuson, et al., 2015). Early experiences and performance in mathematics even yield additional effects well beyond classrooms, with consequences affecting economic prosperity, well-being, and quality of life (Campbell, Conti, Heckman, Moon, & Pinto, 2013; Clements & Sarama, 2016; Denton, & Flanagan, & West, 2002). "Providing young children with extensive, high-quality early mathematics instruction can serve as a sound foundation for later learning in mathematics and contribute to addressing long-term systematic inequities in educational outcomes" (Cross et al., 2009, p. 2).

In *Catalyzing Change in Early Childhood and Elementary Mathematics: Initiating Critical Conversations,* NCTM (2020) calls for mathematics education system that is just, equitable, and inclusive for each and every child—and for the self-reflections, collaborative actions, and critical conversations this requires among all stakeholders. American students today bring with them to school a wide array of cultural and linguistic backgrounds, prior knowledge, readiness, interests, motivations, home situations, and learning styles. While it is critical that all students have high expectations for learning as well as access to high-quality instruction, it is also necessary that schools provide differentiated supports that address and leverage each student's cognitive and affective needs and assets if mathematics success is to be achieved broadly (Gay, 2013; Gutiérrez, 2013; Hammond, 2014; Ladson-Billings, 1995, 1997, 2021, Larson, 2018; Lawrence-Brown, 2004; NCTM, 2014, 2015, 2020; Tomlinson, 1997, 2017; Turner et al., 2013). "[W]e embrace a perspective on equity that supports teaching practices and reflective tools focused on empowerment of the whole child…. All students, in light of their humanity—their personal experiences, backgrounds, histories, languages, and physical and emotional well-being—must have the opportunity and support to learn rich mathematics that fosters meaning-making, empowers decision-making, and critiques, challenges, and transforms inequities and injustices. Equity does not mean that every student should receive identical instruction. Instead, equity demands that responsive accommodations be made as needed to promote equitable access, attainment, and advancement in mathematics education for each student" (Aguirre et al., 2013, p. 9).

Math Expressions©2026 has a global research base, proven at successfully supporting math learning across cultures and contexts. Via its Class Learning Path approach, the program places a strong emphasis on differentiation for teaching all learners and on intervention in response to specific needs. Additionally, the program's emphasis on the visual (such as through Math Drawings) and the verbal (such as through Math Talk) varied and supportive points of entry and progressive pathways for diverse groups of learners to develop understanding and collaboratively co-construct solutions to math tasks and problems.

Promoting Equity, Access, and Rigor

"Creating, supporting, and sustaining a culture of access and equity require being responsive to students' backgrounds, experiences, cultural perspectives, traditions, and knowledge when designing and implementing a mathematics program and assessing its effectiveness. Acknowledging and addressing factors that contribute to differential outcomes among groups of students are critical to ensuring that all students routinely have opportunities to experience high-quality mathematics instruction, learn challenging mathematics content, and receive the support necessary to be successful. Addressing equity and access includes both ensuring that all students attain mathematics proficiency and increasing the numbers of students from all racial, ethnic, linguistic, gender, and socioeconomic groups who attain the highest levels of mathematics achievement" (NCTM position statement on Access and Equity in Mathematics Education, 2015, online).

While mathematics achievement on every scale requires that all students be expected to meet rigorous standards, each student comes to school with a unique background, skill set, perspective, and set of strengths and needs—and therefore must receive effective, individualized support to realize and enjoy success in math learning (Clements & Sarama, 2020; NCTM, 2014, 2015, 2020; National Mathematics Advisory Panel, 2008; Shapka, Domene, & Keating, 2008). Despite the obstacles that some children, particularly those from historically underserved populations, endure, all children are capable of learning and performing in math at high levels. Research has documented that significant positive outcomes are possible when schools and teachers address issues of equity and access within mathematics instruction (Gutiérrez, 2013; Kisker, Lipka, Adams, Rickard, Andrew-Ihrke, Yanez, & Millard, 2012; Lawrence-Brown, 2004; McKenzie, Skrla, Scheurich, Rice, & Hawes, 2011; Schoenfeld, 2002). “Just, equitable, and inclusive learning opportunities for all students demand change in institutional structures, teaching, and learning environments, and individual beliefs and actions" (NCTM, 2020, p. 25).

For over two decades, the National Council of Teachers of Mathematics has advocated for more equitable practices to close learning gaps and ensure all students succeed in learning math. “All students, regardless of their personal characteristics, backgrounds, or physical challenges, must have opportunities to study—and support to learn—mathematics. Equity does not mean that every student should receive identical instruction; instead, it demands that reasonable and appropriate accommodations be made as needed to promote access and attainment for all students” (2000, p. 12). In its 2015 position statement on Access and Equity in Mathematics Education, NCTM also urges broad attitudinal shifts on the part of educators:

> To close existing learning gaps, educators at all levels must work to achieve equity with respect to student learning outcomes. A firm commitment to this work requires that all educators operate on the belief that all students can learn. To increase opportunities to learn, educators at all levels must focus on ensuring that all students have access to high-quality instruction, challenging curriculum, innovative technology, exciting extracurricular offerings, and the differentiated supports and enrichment necessary to promote students' success at continually

> advancing levels. Providing all students with access is not enough; educators must have the knowledge, skills, and disposition necessary to support effective, equitable mathematics teaching and learning. (online)

It is vital that educators understand that long-standing achievement gaps have been caused by pervasive inequalities that have historically afforded significantly fewer resources and opportunities to certain groups (Aronson & Laughter, 2016; Aguirre et al., 2013; Cross et al., 2009; Flores, 2007; Gutiérrez, 2013; Lawrence-Brown, 2004; NCTM, 2014; Ukpokodu, 2011). To address issues of equity and access, calls are increasing for educators to shift away from perceptions that students from historically disadvantaged backgrounds are deficient; rather, educators are encouraged to adopt a culturally responsive approach in which the distinct cultural, linguistic, and environmental experiences that students bring to school are viewed as assets to be respected, embraced, and leveraged to optimize learning for individual students as well as their peers (Aguirre et al., 2013; Flores, 2007; Gay, 2018; Gutiérrez, 2013; Hammond, 2014; Ladson-Billings, 1995, 1997, 2021; Lawrence-Brown, 2004; NCTM, 2014, 2015, 2020; Paris, 2012; Ukpokodu, 2011; Xenofontos, 2019). For ELs specifically, classroom Math Talks must be structured via protocols, practicing, and positioning so as to ensure the active, agentic participation of all students, wherever each is in their language development (Turner et al., 2013).

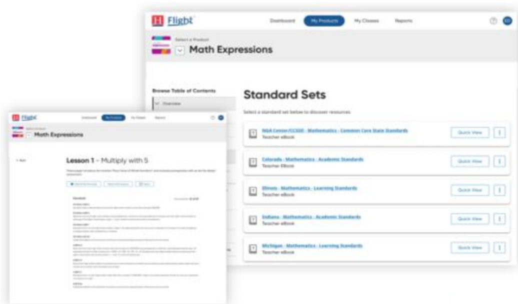
NCTM (2014, 2015) calls for the following practices to address access and equity in mathematics education and thereby improve student outcomes and achievement on a wide range of measures, including assessments, dispositions, and persistence transcendent of students' racial, ethnic, linguistic, gender, and socioeconomic statuses:

- implementing rigorous standards and holding high expectations for all learners who additionally have access to high-quality mathematics curricula and instruction taught by skilled and effective instruction
- allowing adequate time for students to learn
- placing appropriate emphasis on differentiated processes that broaden students' productive engagement with mathematics and that support individual students as needed and appropriate
- monitoring student progress and making accommodations, interventions accordingly
- leveraging strategic use of human and material resources

As Vygotsky (1978) noted in his seminal research on learning, “Optimal learning takes place within students’ ‘zones of proximal development’—when teachers assess students’ current understanding and teach new concepts, skills, and strategies at an according level” (p. 86). Differentiated instruction is a well-established, evidence-based, equity-driven practice of flexibly altering teaching that maximizes learning for all students and yields positive outcomes across achievement levels, including specifically within mathematics. A differentiated approach to instruction recognizes and supports the classroom as an inclusive community where students are nourished as individual learners and provided with an appropriate, motivating balance of challenge and success. In effective differentiated environments, instruction is responsive to students' varying readiness, interests, and learning profiles and, accordingly, offers students varying levels of assessment-based expectations for task completion within a lesson or unit (Chamberlin & Powers, 2010; Kalbfleisch & Tomlinson, 1998; Lawrence-Brown, 2004; National Mathematics Advisory Panel, 2008; Rogers & Garii, 2013; Tomlinson, 1997, 2001, 2004, & 2017). "When teachers differentiate how they support student learning, they are ensuring that all students have access to the content and skills that have been identified in the standards and that educational opportunities are equitable for all students" (Rogers & Garii, 2013, p. 34-35).

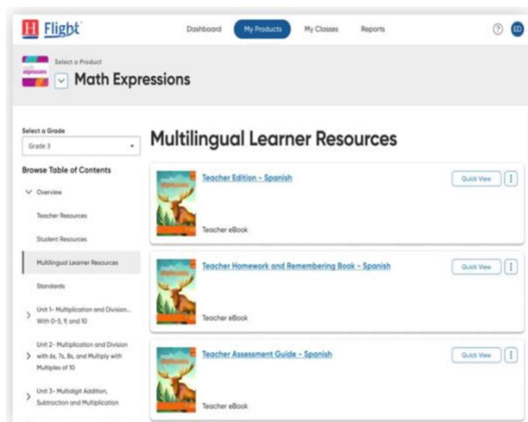
How *Math Expressions* Delivers

Augmented with expanded resources available via ***Matific*** and **Heinemann *Flight***, *Math Expressions* offers rigorous standards-aligned content along with differentiated scaffolded supports to promote a growth mindset and optimize the rich potential of each and every learner across diverse classrooms.



The student experience within this suite of resources features simplified interface for better usability and vibrant, user-friendly interactivity with digital access to: **Lessons & Activities; Games & Practice; Assignments & Assessments; and Adaptive & Differentiated Learning**. These features elevate learning and boost student engagement while providing each learner with an individualized pathway to success.

As cited above, a long-established body of research demonstrates the effectiveness of *Math Expressions* in supporting both the mathematical and language development of English learners, particularly within the verbally rich and collaborative Math Talk Communities the program fosters. The further meet the needs and leverage assets of ELs, available via the *Heinemann Flight* online platform is an array of **Multilingual Learner Resources**, which include lessons, activities, and assessments and Spanish-language materials.



Distributed Practice

Among math educators in the United States, a historical tension has existed between understanding and fluency; on one side is an emphasis on exploration facilitated by sensory experiences with objects and open-ended activity, and on the other is a focus on rote practice and worksheets without attention to the construction of meaning. Fuson (2009) stresses that students learning math are best served by a balanced approach that is child-centered within a teacher-guided structure based on individualized pathways driven by students' progress and needs—featuring a dual focus on both understanding and fluency. "Procedural knowledge and conceptual understandings must be closely linked" (NRC, 2005, p. 232) and effective mathematics cannot have one without the other, for concepts and procedures develop in tandem and iteratively, with gains in one supporting gains in the other (NCTM, 2014; Rittle-Johnson, Schneider, & Star, 2015).

Conceptual understanding is knowledge of abstract and general principles, whereas procedural understanding is knowledge of the steps or actions between a goal that is then applied with varying degrees of fluency (Rittle-Johnson et al., 2015). Procedural fluency is a critical component of mathematical proficiency. However procedural fluency is about more than memorizing facts or steps; it entails the following capacities: to apply procedures accurately, efficiently, and flexibly; to build and modify procedures as well as transfer them to different problems and contexts; and to recognize when one strategy or procedure is more appropriate to apply than another. In developing procedural fluency, students need experience integrating concepts and procedures and understanding patterns among them. Students also need to build on familiar procedures in the process of creating their own informal strategies and procedures via opportunities to support and justify their choices of appropriate procedures. To be mathematically proficient, students need a deep and flexible knowledge of a variety of procedures, along with an ability to make critical judgments about which procedures or strategies are appropriate for use in particular situations—and the goal for students developing procedural fluency is to acquire a body of known facts and generalizable methods that will allow them to efficiently and accurately solve varied problems (Baroody, 2006; Kling & Bay-Williams, 2014; NCTM, 2014; NRC, 2001, 2005, & 2012). When students are able to connect procedures and concepts, and when learning is meaningful, retention improves and students are better able to apply what they know in different situations. If students memorize and practice procedures without conceptual understanding, they lack capacity to apply procedures and the motivation to use them effectively (Fuson et al., 2005; Hiebert, 1999). "Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems" (NCTM, 2014, p. 42).

Research has firmly established that to develop students' procedural fluency, teachers should build on a foundation of conceptual understanding; support students in looking for patterns; allow students to flexibly choose among solution methods; and offer targeted opportunities for purposeful, meaningful practice that are brief and distributed over time, rather than rote and repetitive (Baroody, 2006; Fuson & Beckmann, 2012; Fuson et al., 2005; Fuson & Murata, 2007; NCTM, 2014; Rohrer, 2009). "Practice does not mean rote experiences.

Practice involves repeated experiences that give children time and opportunity to build their ideas, develop understanding, and increase fluency (Fuson, et al., 2015, p. 64).

Studies have contrasted “massed practice” and “distributed practice,” with distributed practice generally found to be better than massed practice for retention (e.g., Underwood, 1961; Cepeda, Pashler, Vul, Wixted, Rohrer, 2006), particularly with practice closer together just after learning and then becoming spaced farther apart (Carpenter, Cepeda, Rohrer, Kang, & Pashler, 2012). Practice sessions should also interweave worked example solutions with independent problem-solving, and as students gain proficiency, the number of worked examples should be decreased while the number of independently solved problems increases (Pashler, Bain, Bottge, Graesser, Koedinger, McDaniel, & Metcalfe, 2007). Rather than assigning students a greater number of problems, explaining worked examples, both correct and incorrect, have also been shown to foster deeper conceptual understanding and to be effective in helping students learn to solve problems faster—perhaps because these worked problems help to reduce students' cognitive loads and allow them to focus on the targeted learning (Booth, Lange, Koedinger, & Newton (2013).

After students have established a strong conceptual foundation and the ability to explain the mathematical basis for a procedure or strategy, they should practice with a small number of carefully selected problems and receive ongoing feedback on their progress (Rohrer, 2009). If students have memorized and practiced procedures that they do not understand, they have less motivation to understand their meaning or the reasoning behind them (Hiebert, 1999). When learning is not meaningful and is disconnected from other knowledge, students have more difficulty absorbing concepts; when students are able to connect procedures and concepts, their retention improves and they are better able to apply what they know in different situations (Fuson et al., 2005).

Worth noting is that not everything taught in mathematics fits neatly into a conceptual progression. While there is a temptation “to want to discover universal progressions in learning that are driven by deep changes in conceptual structure . . . there are parts of mathematics learning that, although important and complex, are driven by more incremental mechanisms...This does not suggest, however, that isolated, repeated practice is effective, but rather than there are some mathematical skills which may be best developed with practice in the context of a “meaningful examination of patterns and strategies” (Sherin & Fuson, 2005, p. 385-386).

How *Math Expressions* Delivers

Within *Math Expressions* **Inquiry Learning Path**, the final phase, **Maintaining and Integrating Fluency**, has teachers giving occasional problems and using instructional discussions and dialog to help students remember what they have learned. With this planned and purposeful assistance, teachers give students the practice and recall they need to make connections between what they know and are about to learn. This phase is earmarked by distributed practice and feedback that relates other mathematics topics when they arise. **Distributed practice** maintains and strengthens understanding of a concept or skill.

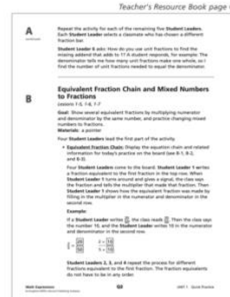
Quick Practice is a crucial part of *Math Expressions* at all grades. Students need to do the **5-minute Quick Practice** every day. Many teachers start the lesson with the Quick Practice while others do it at some other time of the day. The Quick Practices focus on the concepts and skills throughout the year that most need to be brought to mastery by all students. **Quick Practice** routines are a **whole-class activity** that provide opportunities for students to call to mind their **prior understanding** of a topic that has already been discussed in class or to begin to **build prerequisite skills** for a topic that is to come later. They also generate **immediate feedback** at the class and/or individual student level, enabling prompt intervention as necessary. Quick Practices build **classroom belonging** as students do these activities together. The teacher can lead the Quick Practices initially but soon should help **Student Leaders** lead the activities. This provides all students with leadership experiences that allow them to bloom and feel part of the classroom. Many Quick Practices involve **choral responses** that allow students to build feelings of being part of the classroom and allow teachers to monitor understanding and progress in real-time.

Teacher's Resource Book page Q1

A

Smaller and Smaller Unit Fractions and How Much More to 1?

Teacher's Resource Book page Q2



B

Equivalent Fraction Chain and Mixed Numbers to Fractions

The ***Matific*** adaptive digital platform is a rich source of additional interactive and personalized practice with one-on-one alignment with *Math Expressions* lesson content. Matific's game-based format and engaging graphics and interface assure boosted student motivation. *Matific's* activities focus on developing conceptual understanding alongside procedural fluency, reinforcing *Math Expression*s' curriculum. Matific offers **Targeted Skill Development,** providing **differentiated practice** for students struggling with specific concepts and addressing individual learning gaps. Insights from *Matific* inform teacher instruction, allowing for **data-driven, personalized interventions** and further **differentiated learning activities** to support each and every student in meeting rigorous mathematics standards.

Professional Learning

To support the delivery of effective instruction, Heinemann features research-based approaches to professional learning that supports teachers in becoming developers of high-impact learning experiences for their classroom. Through Implementation Success support and personalized Job-Embedded Coaching, Heinemann's experienced Instructional Coaches provide comprehensive professional learning solutions that are data and evidence driven, mapped to instructional goals, and centered on student success. Additionally, these professional learning opportunities aim to build educators' collective capacity focusing on practices for everyday teaching.

Heinemann Instructional Coaching offers ongoing, personalized support focused on high-impact strategies, planning, and achieving classroom goals. Expert Coaches guide teachers through year-long collaboration, modeling, and coaching to facilitate instructional strategies, planning, practices, and the effective implementation of Math Expressions in the classroom.

Connected Professional and Personalized Learning

Effective curriculum-based professional learning consists of ongoing, active experiences that focus on improving the rigor and impact of instructional practices and ideally replicate the learner-centered approaches that teachers are expected to provide for their students. Elements of effective curriculum-based PL include high-quality educative curriculum materials, transformative learning experiences that shift teachers' attitudes, beliefs, and practices, and a prioritization of equity to ensure all students meet high expectations. Functional design elements include learning designs that model inquiry-based instruction, experiences to shift teachers' beliefs, opportunities for reflection and feedback, and change management strategies that address individual concerns and group challenges. Finally, structural design features include collective participation in which teachers practice and reflect on the curriculum, models of learning that evolve from initial use to ongoing support to building capacity, and a considered use of time. These elements of effective curriculum-based PL must exist in a system with strong leadership, adequate resources, and coherence towards common goals (Short & Hirsh, 2020).

How professional learning is delivered has an impact on its effectiveness. PL programs with teacher-to-teacher collaboration focused on instructional improvement – whether in professional learning communities (PLCs), teacher teams, or group work in PL sessions – have demonstrated improvement in teachers' instructional skills. Another effective practice is conducting follow-up meetings or coaching sessions after the initial implementation of a program so that teachers can share their experiences and receive feedback. The content of the PL is equally important. It should focus on subject-specific instructional practices (not merely content knowledge), prioritize specific supportive materials over general principles, and help teachers build stronger relationships with students (Hill & Papay, 2022).

Long-term connected professional learning includes cohesive features—online coaching, observations, and collaboration—all with a focus on how to ensure social and emotional well-being and meaningful student learning in digital environments. A connection between workshops, coaching, and collaboration is essential for a professional learning program to make a difference in student achievement. Connecting workshops, follow-up coaching, and support among peers can help teachers retain new knowledge, practice new skills, and share effective approaches that they can scale (Aguilar, 2019).

Effective professional learning, whether in-person, online, or blended, offers teachers coherent experiences so that their learning is connected to their work in the classroom and builds proficiency. This approach includes alignment between the study of theory and practice, observation of theory and practice, individual coaching, and further practice and refinement through collaboration. Each of these components is essential to support and build on the content and pedagogy that is learned, observed, and practiced in each of the other components (Rock, 2019).

For schools to support the implementation of high-quality instructional materials, effective professional learning during the launch of the curriculum, when teachers are learning and committing to an instructional approach, is critical (Gulamhussein, 2013). Teachers' initial exposure to a concept should engage them through varied approaches and active learning strategies to make sense of the new practice (Bill & Melinda Gates Foundation, 2014; Garet et al., 2001; Gulamhussein, 2013). An effective professional learning program should be curriculum-based and focused on targeted content, strategies, and practices (Bill & Melinda Gates Foundation; 2014; Saxe et al., 2001; Wei, 2009) and be grounded in the teacher's grade level or discipline (Gulamhussein, 2013).

Online PD can help solve resource challenges in implementing a scalable and sustainable model. Online PD platforms can create a peer-to-peer support community, building the capacity of the teaching team to support each other. Perhaps most importantly, online PD allows teachers to experience the agency and personalized learning they are creating for students. The unique opportunity of blended PD is the shift from PD as a one-time or periodic event to PD as an ongoing and embedded practice (Tucker & Wycoff, 2019).

Many school districts and providers of teachers' professional development are moving toward a more personalized model of professional development, taking a cue from the movement toward personalized learning for students. This approach often focuses on short modules, which teachers can choose and then complete on their own time. The modules can incorporate aspects of gamification, micro-credentialing, and online professional development communities. By allowing teachers to choose their own professional development courses and activities and complete them in their own place at their own pace, the professional development will be better matched to their needs. Teachers will be able to set goals, find resources to help them meet those goals, track their progress, and get feedback from supervisors and colleagues (Gamrat et al., 2014; Meeuwse & Mason, 2018).

Providing teachers with time and frameworks to collaborate on improving their instruction, through Professional Learning Communities (PLCs) or Teacher Study Groups (TSGs) has the potential to improve teachers' knowledge, instructional practices, and student achievement. A study of TSGs focused on reading comprehension and vocabulary instruction found that teachers who participated in TSGs saw significant improvements in their knowledge of vocabulary instruction and their teaching practices. Students of the TSG teachers also saw improvements in oral vocabulary (Gersten et. al., 2010).

How *Math Expressions* Delivers

Heinemann provides a continuum of connected professional learning designed to foster teacher agency, promote collaboration, and build collective efficacy and capacity to support teachers' role as designers of quality instruction. Through strategic planning, guided implementation support, and blended coaching. Heinemann helps schools and districts achieve measurable gains with professional learning centered on research-based practices and student outcomes.

Guided Implementation Support Included with *Heinemann Math Expressions*

Guided Implementation Support for *Heinemann Math Expressions* assists educators in building confidence and success with their new Math program. A **Getting Started** session provides learner-centered foundational program knowledge through exploration and hands-on activities. It is the first step toward a successful implementation.

Ongoing training and support resources are also provided through On Demand videos on Flight, Heinemann's learning platform. There, teachers access program-specific bite-sized videos focused on subject-specific support.

Heinemann continues to provide a personalized model for professional development, with live online Follow-Up courses designed to deepen teachers' learning and application through live online sessions. These sessions offer additional support on *Math Expressions* topics like digital tools, resources, assessment, data, and reports, providing additional personalized professional development.

Coaching to Strengthen Teaching and Learning

Research has demonstrated that sustained, job-embedded coaching is the most effective form of professional learning, whether it is delivered in person or online. Coaching delivered in person is most effective when coaches are highly expert and focus their work with teachers on a clearly specified instructional model or program. Other opportunities for teachers to develop their knowledge of the targeted instructional model (e.g. in courses, workshops or coach-led learning groups) are also an important component of successful coaching programs. Online coaching shows promise for being at least as effective as in-person coaching for improving outcomes, though the research base comparing delivery systems is thin. The balance of evidence to date, however, suggests that the medium through which coaching is delivered is less important than the quality and substance of the learning opportunities provided to teachers (Matsumara et al, 2019).

A recent meta-analysis of coaching programs found overall effect sizes of 0.49 SD on instructional practices and 0.18 SD on student achievement. Encouragingly, teachers who received online coaching performed similarly to teachers who received in-person coaching for improving both instructional practices and student achievement. The authors identified several aspects of online coaching as potential strengths: increasing the number of teachers with whom a high-quality coach can work, reducing educators' concern about being evaluated by their coach, and lowering costs while increasing scalability (Kraft, Blazar, & Hogan, 2018).

The best evidence for coaching is found for one-to-one coaching, where coaches observe and offer feedback on teachers' practice. There is not a great deal of research on specific coaching practices, but effective coaches might engage in co-planning, modeling, or guiding teacher reflection. Effective coaching is time-intensive and the most successful programs have invested in the selection, training, and ongoing support of their coaches (Hill & Papay, 2022).

Virtual coaching can provide a framework for a shared leadership structure that focuses on facilitating teachers' autonomy, self-management, empowerment, and cooperation. Because the coach and the teacher jointly pursue the goal of increased student achievement, virtual coaching provides social support for both parties, leading to enhanced emotional and psychological strength. Any coaching relationship—traditional or virtual—builds on several underlying qualities of both teacher and coach. Chief among them are a willingness to change, a trusting relationship, a high level of initiative, and a personal and organizational commitment to the workplace (Blackman, 2010).

How *Math Expressions* Delivers

Heinemann Coaching is grounded in a research-based Coaching Framework that leverages instructional best practices proven to impact student success and effective implementation of *Heinemann Math Expressions* in the classroom. It offers job-embedded, sustainable, data-driven, and personalized support aligned to each teacher's unique learning goals. Available through virtual sessions or in-person coaching days in cohorts between one and twenty, Heinemann Coaching supports every teacher to elevate instructional practice, meet district goals, and raise student achievement.

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